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# Generalized condition-based maintenance optimization for multi-component systems considering stochastic dependency and imperfect maintenance

Jun Xu, Zhenglin Liang *, Yan-Fu Li, Kaibo Wang

*Industrial Engineering Department at Tsinghua University*

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ABSTRACT

With the development of sensor and communication technology, condition-based maintenance (CBM) attracts increasing attention, especially for multi-component systems. This paper aims to investigate the optimal CBM policy under periodic inspection for a *K*-out-of-*N*: G system, where economic dependency, stochastic dependency and imperfect maintenance are emphasized. The objective is to minimize the expected long-run discounted cost. In the model, the cumulative degradation of each component is modeled by heterogeneous stochastic processes, the dependence among all components is characterized by a copula function, and the imperfect maintenance is represented by a reduction in the degradation level. Since the system has Markov property, we solve the CBM optimization problem based on Markov decision process (MDP) framework. To ease the computation burden, we discretize the continuous state space and then use the value iteration algorithm with Monte Carlo simulation to find the optimal inspection interval and the optimal CBM policy. Numerical studies for a 1-out-of-2: G system are conducted to systematically examine the impacts of degradation processes, copula functions and imperfect maintenance on the optimal maintenance decisions, which provides insights for multi-component system maintenance. A sensitivity analysis of cost-related parameters is also performed.

## 1. Introduction

Maintenance is important for managing system reliability, preventing system failures and improving the effectiveness of system operations. Maintenance policies can be generally classified as Time-Based Maintenance (TBM) and Condition-Based Maintenance (CBM) policies, and in both categories, one can deal with Corrective Maintenance (CM) and Preventive Maintenance (PM) [1]. TBM is scheduled based on a specific lifetime model over elapsed time, while CBM makes use of both degradation models and collected sensor data that can represent the system health conditions.

To perform CBM, the first critical step is to properly model the degradation process. Systems with discrete-state degradation are usually modeled by Markov processes. The corresponding CBM problems are formulated as Markov decision processes (MDPs) or its variants and then solved by standard algorithms. With the enrichment of the degradation data collected from sensors, we are able to model the degradation process with continuous states, which can be more accurate and flexible. For such degrading systems, Wiener process, Gamma process and Inverse Gaussian (IG) process [2,3] are three commonly used stochastic degradation models. For example, Wiener process and IG process have been applied to GaAs laser degradation and fatigue crack growth [4,5], and Gamma process has been used to model the corrosion of feeder pipes [6] in practice. In this study, all the above three stochastic processes are investigated and we consider the heterogeneous case.

Existing CBM literatures mainly focus on single-component systems, because for multi-component systems, the probability analysis and the optimal maintenance decisions are much more complex [7]. If there is only one critical component in the system, or if a one-dimensional "health index" is used, the CBM policy for the single-component system may be applicable to the multi-component system [8]. However, many systems in practice contain multiple critical components and using a one-dimensional "health index" is insufficient. Thus, it is necessary to develop proper CBM policies for multi-component systems, where various types of dependency among components need to be considered [9]. First, economic dependency indicates that the costs can be reduced when several components are maintained together. Second, structural dependency indicates that some working components have to be replaced or removed in order to repair some failed components. Third,

* Corresponding author.
*E-mail address:* zhenglinliang@tsinghua.edu.cn (Z. Liang).

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**Notations**

| | |
|---|---|
| $K$ | Minimum number of working components for the system to work |
| $N$ | Number of components in the system |
| $Y_i(t)$ | Degradation level of component $i$ at time $t$ |
| $\boldsymbol{Y}(t)$ | Vector of the degradation levels of all components at time $t$ |
| $dr_i$ | Mean degradation rate of the degradation process of component $i$ |
| $L_i$ | Predetermined failure threshold for component $i$ |
| $T_i$ | Failure time of component $i$ |
| $T$ | Failure time of the system |
| $\mathscr{C}(\cdot)$ | Copula function |
| $\mathscr{I}(\cdot)$ | Independence copula |
| $\tau$ | Kendall's tau for a bivariate copula |
| $Y_i(t_0^-)$ | Degradation level of component $i$ immediately before PM at time $t_0$ |
| $PM_{i,Y_i(t_0^-)}$ | Reduction in the degradation level when performing PM on component $i$ at time $t_0$ |
| $p_i$ | Probability that component $i$ after PM is as-good-as-new |
| $\delta$ | Inspection interval |
| $c_i$ | Inspection cost |
| $c_c$ | CM cost |
| $c_d$ | Downtime cost per time unit |
| $c_s$ | Shared setup cost for PM |
| $c_p$ | PM cost for one component |
| $r$ | Discount factor |
| $I_\delta$ | Expected long-run discounted inspection cost under $\delta$ |
| $y_i$ | Degradation level of component $i$ upon inspection |
| $\mathbf{y}$ | Vector of $y_i$ |
| $V_\delta(\mathbf{y})$ | Expected long-run discounted maintenance cost in state $\mathbf{y}$ under $\delta$ |
| $S$ | Set of all degradation states |
| $S^F$ | Set of all system failure states |
| $A(\mathbf{y})$ | Set of all actions available in state $\mathbf{y}$ |
| $\pi_\delta(\mathbf{y})$ | Action taken in state $\mathbf{y}$ under deterministic CBM policy $\pi_\delta$ |
| $y_i^A$ | Degradation level of component $i$ immediately after taking action |
| $\mathbf{y}^A$ | Vector of $y_i^A$ |
| $y_i^{'}$ | Degradation level of component $i$ upon the next inspection |
| $\mathbf{y}^{'}$ | Vector of $y_i^{'}$ |
| $C(\mathbf{y}, a, \mathbf{y}^A, \mathbf{y}^{'})$ | Expected cost of transition from state $\mathbf{y}$ to $\mathbf{y}^A$ after taking action $a$, and finally to $\mathbf{y}^{'}$ |
| $D_\delta(\mathbf{y}^A, \mathbf{y}^{'})$ | Expected downtime cost in state $\mathbf{y}^A$ and transition to state $\mathbf{y}^{'}$ under $\delta$ |
| $D_\delta(\mathbf{y}^A)$ | Expected downtime cost between two successive inspection epochs in state $\mathbf{y}^A$ under $\delta$ |
| $T_{\mathbf{y}^A}$ | Failure time of the system with initial state $\mathbf{y}^A$ |
| $Q_\delta(\mathbf{y}, a)$ | Expected long-run discounted maintenance cost in state $\mathbf{y}$ taking action $a$ under $\delta$ |

stochastic dependency indicates that the failure or deterioration of one component can affect the degradation processes of other components. In this study, the economic dependency is incorporated in the optimization objective and the stochastic dependency is specially emphasized.

To model the stochastic dependency, researchers have focused on different aspects, such as the failure dependence [10], degradation dependence [7], degradation rate dependence [9,11], and shock dependence [12]. In this study, we use a copula function to characterize the dependence structure among all components. Copula function has been widely used in modeling failure dependence [10,13] and degradation dependence [14,15] in recent literatures. For example, Safaei et al. [13] investigated the age replacement policy for repairable series and parallel systems with $n$ dependent components by using three types of copula function to model the dependence structure among the lifetime distributions. Liu et al. [15] proposed a life cycle model for systems subject to multiple dependent Gamma processes and considered the environmental influence on the degradation rate and failure threshold, where Clayton copula was used to characterize the dependence among degradation increments. In addition, Mireh et al. [16] considered the dependence between the Gamma degradation process and Weibull distributed hard failure time by Frank copula function.

Based on the identified system degradation model, the optimal maintenance decisions can be constructed according to certain criteria (such as cost, availability, reliability, etc.). The optimal decisions often involve two aspects: inspection schedule and maintenance policy. The majority of CBM problems assume perfect inspection with continuous, periodic or non-periodic review, and sometimes imperfect inspection may be more realistic because there may exist inspection error [17]. As for maintenance policy, opportunistic maintenance policy [18,19] and dynamic grouping maintenance policy [20,21] are two popular CBM policies for multi-component systems. Under opportunistic maintenance policy, the maintenance of some components may provide an opportunity for the maintenance of the remaining components, which can reduce the maintenance cost. Under dynamic grouping maintenance policy, components in the system are partitioned into several groups, where the maintenance is performed on all the components in one group to save setup cost. Genetic algorithm [22] and dynamic programming [23] are also applied to find the optimal maintenance policy.

The maintenance policy is influenced by the degree of maintenance as well. The perfect maintenance is easy to analyze, but most PM activities in real world are imperfect. The imperfect maintenance restores the health condition of a degrading system to any degradation level between as-good-as-new and as-bad-as-old [24]. Recently, Sun et al. [25] investigated the optimal CBM strategy for a $K$-out-of-$N$: G system with $N$ identical and independent components under periodic inspection. In their model, the degradation of each component was assumed to follow a Wiener process and Markov decision framework was used to find the optimal inspection interval and maintenance strategy to minimize the total cost. Inspired by the approach of Sun et al. [25], we make efforts to bring the model closer to reality by emphasizing the stochastic dependency among all components and considering imperfect maintenance.

In this study, we aim to investigate the optimal CBM policy under periodic inspection for a $K$-out-of-$N$: G system with imperfect maintenance. Comparing to the extant literature, the main contributions are as follows: (1) we propose a CBM modeling framework that incorporates economic dependency, stochastic dependency and imperfect maintenance, which can be extended to other multi-component maintenance problems; (2) we utilize the MDP framework with Monte Carlo simulation to find the optimal maintenance decisions of $K$-out-of-$N$: G systems with stochasticity, heterogeneity, and dependency; (3) we systematically examine the impacts of degradation processes (including the non-monotonic Wiener process and monotonic Gamma process and IG process) and copula functions on the optimal maintenance decisions; (4) we investigate the impacts of imperfect maintenance on the optimal maintenance decisions and conduct comprehensive sensitivity analysis.

The rest of the paper is organized as follows: In Section 2, we detail the multi-component degrading system and the maintenance settings by emphasizing stochastic dependency and imperfect maintenance. In Section 3, we utilize the MDP based method to solve the proposed CBM optimization problem. In Section 4, numerical studies for a 1-out-of-2: G system are conducted to systematically examine the impacts of degradation processes, copula functions and imperfect maintenance on the optimal maintenance decisions, and the application of the proposed framework in general $K$-out-of-$N$: G systems is illustrated with a 2-out-of-3: G system. Sensitivity analysis on cost-related parameters is also carried out. Finally, we conclude this study and discuss some future research directions in Section 5.

## 2. System degradation modeling

### 2.1. Basic system description and assumptions

The system under study is a $K$-out-of-$N$: G system with $N$ dependent components subject to degradation. Let $Y_i(t),\ i = 1, 2, \ldots, N$ denote the degradation level of component $i$ at time $t$. Typically, the initial degradation level for a brand new component is zero, that is, $Y_i(0) = 0,\ i = 1, 2, \ldots, N$. The system will fail if less than $K$ components work in normal conditions. Periodic inspections are carried out with interval $\delta$ and cost $c_i$. If the system is found failed, CM is performed immediately with CM cost $c_c$ and downtime cost $c_d$ per time unit. Otherwise, a CBM policy is used to decide which components to be maintained by PM: If PM is performed on $n$ components, $n = 1,2,\ldots,N$, a shared setup cost $c_s$ and PM cost $c_p$ for each component will be incurred, which is $c_s + nc_p$ totally. The overall assumptions are listed below.

**Assumptions about degradation processes**

**A1.** Component $i$ fails if its degradation level reaches a predetermined failure threshold $L_i$, i.e., $Y_i(t) \geq L_i,\ i = 1, 2, \ldots, N$.

**A2.** The degradation process for each component follows a stochastic process with independent increments. Wiener process, Gamma process and IG process are considered in this study, as they are commonly used in degradation modeling and have complementary properties [3]. Wiener process is non-monotonic and can represent systems that have a self-recovery nature, while Gamma process and IG process are monotone. Please see Section 2.2 for a brief introduction.

**A3.** Degradation dependence can be characterized by a copula function. Copula is widely used for representing multiple dependent stochastic processes in system engineering. Please see Section 2.3 for details.

**Assumptions about maintenance actions**

**A4.** The conditions of the studied system are not self-announcing. As a result, component failure and system failure can only be detected upon inspection.

**A5.** The time for inspections and the time for maintenance actions are negligible comparing with the degradation process and the interval between two successive inspections.

**A6.** CM is perfect. CM here is equivalent to the replacement of the system. When the system fails, all remaining components are also seriously damaged that are cost-ineffective/unable to be restored by PM. Thus, an entire replacement is required. When the system remains operating, the failure of components has less effect on other components. This dependence can be modeled as a copula function.

**A7.** PM is imperfect, the damage reduction of imperfect maintenance is represented by a random variable and the damage reductions of different components are independent. On the one hand, imperfect maintenance is more general in real-world, under which it is nearly impossible to restore the component as-good-as-new. On the other hand, there are many uncertainties in PM, such as the various causes of failure, the proficiency of maintenance workers, the fluctuation of maintenance tools and equipment, and the complex ambient conditions, etc. As a result, even if the degradation levels of several components before PM are identical, their degradation reductions after PM are not necessarily the same. Therefore, it is reasonable to model the degradation reduction of imperfect maintenance by a random variable. Please see Section 2.4 for details.

**A8.** We further assume that the maximum PM cost (i.e., PM is performed on all $N$ components when the system is normal) is less than the CM cost: $c_s + Nc_p < c_c$. This assumption is reasonable because the replacement of a system (CM) is usually more expensive than PM.

Based on the above system configurations, the optimization objective is to find the optimal inspection interval $\delta^*$ as well as the optimal CBM policy to minimize the economic criterion.

### 2.2. Cumulative degradation process

For an individual component, we model its cumulative degradation over time by a stochastic process with independent increments. Let $\{Y_i(t), t \geq 0\}$ denote the cumulative degradation process of component $i$ with $Y_i(0) = 0$. Let $\Delta Y_i(t;s) = Y_i(s+t) - Y_i(s)$ denote the degradation increment from time $s$ to $s+t$, with mean $\text{Mean}_i(t)$, variance $\text{Variance}_i(t)$ and mean degradation rate $dr_i = \text{dMean}_i(t)/dt$ for $i = 1, 2, \ldots, N$ and $s, t \geq 0$. According to Assumption A1, the failure time of each component is characterized by the first hitting time (FHT). FHT $T_i$ for the $i$th degradation process $\{Y_i(t), t \geq 0\},\ Y_i(0) = 0$ to reach its predetermined failure threshold $L_i$ is defined as

$$T_i = \inf_t \{Y_i(t) \geq L_i\},\ i = 1, 2, \ldots, N \tag{2-1}$$

Wiener process, Gamma process and IG process are considered in this study, and their definitions and some properties are given below:

- **Wiener process:** Wiener process is often expressed as

$$Y_i(t) = \mu_i t + \sigma_i B_i(t) \tag{2-2}$$

where $\mu_i$ is the drift parameter, $\sigma_i$ is the diffusion parameter and $B_i(t)$ is the standard Brownian motion. The increment is normally distributed as $\Delta Y_i(t;s) \sim \mathcal{N}(\mu_i t,\ \sigma_i^2 t)$, with $\text{Mean}_i(t) = \mu_i t$, $\text{Variance}_i(t) = \sigma_i^2 t$ and $dr_i = \mu_i$. It has been proved that $T_i$ obeys an IG distribution $\mathcal{IG}\left(\frac{L_i}{\mu_i}, \frac{L_i^2}{\sigma_i^2}\right)$. The probability density function (pdf) and cumulative distribution function (cdf) of $\mathcal{IG}(a, b), b > 0$ are

$$f_{\mathcal{IG}}(y; a, b) = \left(\frac{b}{2\pi y^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{b(y-a)^2}{2a^2 y}\right\}, y > 0 \tag{2-3}$$

$$F_{\mathcal{IG}}(y;a,b) = \Phi\left(\sqrt{\frac{b}{y}}\left(\frac{y}{a} - 1\right)\right) + \exp\left(\frac{2b}{a}\right)\Phi\left(-\sqrt{\frac{b}{y}}\left(\frac{y}{a} + 1\right)\right), y > 0 \tag{2-4}$$

where $a, b$ are the mean and shape parameter and $\Phi(\cdot)$ is the cdf of $\mathcal{N}(0, 1)$. Thus, the cdf of $T_i$ is

$$F_{T_i}(t) = F_{\mathcal{IG}}\left(t; \frac{L_i}{\mu_i}, \frac{L_i^2}{\sigma_i^2}\right), t > 0 \tag{2-5}$$

- **Gamma process:** The increment $\Delta Y_i(t;s)$ follows Gamma distribution $\mathcal{G}\alpha(\alpha_i t, \beta_i)$, with $\text{Mean}_i(t) = \alpha_i \beta_i^{-1} t$, $\text{Variance}_i(t) = \alpha_i \beta_i^{-2} t$ and $dr_i = \alpha_i \beta_i^{-1}$. The pdf and cdf of $\mathcal{G}\alpha(a, b), a, b > 0$ are given by

$$f_{\mathcal{G}\alpha}(y; a, b) = \frac{b^a}{\Gamma(a)} y^{a-1} e^{-by}, y > 0 \tag{2-6}$$

**Table 1**
Kendall's tau ($\tau$), lower tail dependence ($\lambda_L$) and upper tail dependence ($\lambda_U$) for Clayton copula, Frank copula, Gumbel copula, normal copula and t copula.

| | Parameter | $\tau$ | $\lambda_L$ | $\lambda_U$ |
|---|---|---|---|---|
| Clayton copula | $\theta\in(0,\infty)$ | $\frac{\theta}{\theta+2}$ | $2^{-\frac{1}{\theta}}$ | 0 |
| Frank copula | $\theta\in(0,\infty)$ | $1+\frac{4\left(\frac{1}{\theta}\int_0^\theta\frac{t}{e^t-1}\mathrm{d}t-1\right)}{\theta}$ | 0 | 0 |
| Gumbel copula | $\theta\in[1,\infty)$ | $\frac{\theta-1}{\theta}$ | 0 | $2-2^{\frac{1}{\theta}}$ |
| Normal copula | $r_{12}\in[-1,1]$ | $\frac{2}{\pi}\arcsin r_{12}$ | $\mathbb{I}_{\{r_{12}=1\}}$ | $\mathbb{I}_{\{r_{12}=1\}}$ |
| t copula | $r_{12}\in[-1,1]$ $\nu>0$ | $\frac{2}{\pi}\arcsin r_{12}$ | $2F_\nu\left(-\sqrt{\frac{(\nu+1)(1-r_{12})}{1+r_{12}}}\right)$ | $\lambda_U=\lambda_L$ |

$$F_{\mathscr{G}a}(y;a,b)=\frac{\gamma(a,by)}{\Gamma(a)},y>0 \tag{2-7}$$

where $a$ is the shape parameter, $b$ is the scale parameter, $\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}\mathrm{d}x, a>0$ is the Gamma function and $\gamma(a,y)=\int_0^y x^{a-1}e^{-x}\mathrm{d}x, a>0,y>0$ is the lower incomplete Gamma function.

- **IG process:** The increment follows IG distribution as $\Delta Y_i(t;s)\sim\mathscr{IG}(\nu_i t,\lambda_i t^2)$, with $\mathrm{Mean}_i(t)=\nu_i t$, $\mathrm{Variance}_i(t)=\nu_i^3\lambda_i^{-1}t$ and $dr_i=\nu_i$.
For the monotonic Gamma process and IG process, component $i$ fails when $Y_i(t)\geq L_i$. Thus $Y_i(T_i)=L_i,\ i=1,2,\ldots,N$. Note that $\Delta Y_i(t;0)=Y_i(t)-Y_i(0)$ when $Y_i(0)=0$, as a result, the cdf of $T_i$ is

$$\begin{aligned}F_{T_i}(t)&=P(T_i\leq t)=P(Y_i(T_i)\leq Y_i(t))=P(Y_i(t)\geq L_i)=P(\Delta Y_i(t;0)\geq L_i)\\&=1-F_{\Delta Y_i(t;0)}(L_i)\end{aligned} \tag{2-8}$$

where $F_{\Delta Y_i(t;0)}(\cdot)$ is the cdf of $\Delta Y_i(t;0)$ and $F_{\Delta Y_i(t;0)}(\cdot)=F_{\mathscr{G}a}(\cdot;\alpha_i t,\beta_i)$ for Gamma process and $F_{\Delta Y_i(t;0)}(\cdot)=F_{\mathscr{IG}}(\cdot;\nu_i t,\lambda_i t^2)$ for IG process.

### 2.3. Dependent degradation processes

In the realm of system engineering, copula is one of the most prevalent approach for representing multiple dependent stochastic processes. Generally, a copula function is a multivariate distribution function of $N$ standard uniformly distributed random variables [13,26,27]. For an $N$ dimensional random vector $\boldsymbol{X}=[X_1,X_2,\ldots,X_N]$ with marginal cdfs $F_{X_1}(\cdot),F_{X_2}(\cdot),\ldots,F_{X_N}(\cdot)$ and joint cdf $F_{\boldsymbol{X}}(\cdot)$, according to Sklar's theorem, there exists a copula function $\mathscr{C}:[0,1]^N\rightarrow[0,1]$ such that

$$F_{\boldsymbol{X}}(x_1,x_2,\ldots,x_N)=\mathscr{C}(F_{X_1}(x_1),F_{X_2}(x_2),\ldots,F_{X_N}(x_N)) \tag{2-9}$$

and if $F_{X_i}(\cdot),i=1,2,\ldots,N$ are all continuous, then $\mathscr{C}$ is unique [26]. The joint pdf of $\boldsymbol{X}$ is derived as

$$f_{\boldsymbol{X}}(x_1,x_2,\ldots,x_N)=\frac{\partial\mathscr{C}(F_{X_1}(x_1),F_{X_2}(x_2),\ldots,F_{X_N}(x_N))}{\partial F_{X_1}\partial F_{X_2}\cdots\partial F_{X_N}}f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_N}(x_N) \tag{2-10}$$

where $f_{X_i}(\cdot)$ is the marginal pdf of $X_i,i=1,2,\ldots,N$.

To characterize the dependence among all components using copula function, there are two alternatives: (1) to model the dependence among the degradation levels (called degradation levels dependence), and (2) to model the dependence among the degradation increments (called degradation increments dependence).

For the degradation levels dependence, let $\boldsymbol{Y}(t)=[Y_1(t),Y_2(t),\ldots,Y_N(t)]$ be the random vector representing the degradation levels of all $N$ components at time $t$, then the joint cdf of $\boldsymbol{Y}(t)$ can be expressed with a copula function $\mathscr{C}_t(\cdot)$ as

$$F_{\boldsymbol{Y}(t)}(y_1,y_2,\ldots,y_N)=\mathscr{C}_t(F_{Y_1(t)}(y_1),F_{Y_2(t)}(y_2),\ldots,F_{Y_N(t)}(y_N)) \tag{2-11}$$

where $F_{\boldsymbol{Y}(t)}(\cdot)$ is the joint cdf of $\boldsymbol{Y}(t)$ and $F_{Y_i(t)}(\cdot),\ i=1,2,\ldots,N$ are the marginal cdfs of $\boldsymbol{Y}(t)$.

In this study, we focus on the degradation increments dependence, as it is more flexible than the first alternative and is convenient for simulation. Let $\Delta\boldsymbol{Y}(t;s)=[\Delta Y_1(t;s),\Delta Y_2(t;s),\ldots,\Delta Y_N(t;s)]$ (recall that $\Delta Y_i(t;s)=Y_i(s+t)-Y_i(s),i=1,2,\ldots,N)$ be the random vector representing the degradation increments of $N$ components from time $s$ to time $s+t$ for $s,t\geq0$. Then the joint cdf of $\Delta\boldsymbol{Y}(t;s)$ can be computed by

$$F_{\Delta\boldsymbol{Y}(t;s)}(y_1,y_2,\ldots,y_N)=\mathscr{C}_t\left(F_{\Delta Y_1(t;s)}(y_1),F_{\Delta Y_2(t;s)}(y_2),\ldots,F_{\Delta Y_N(t;s)}(y_N)\right) \tag{2-12}$$

where $F_{\Delta\boldsymbol{Y}(t;s)}(\cdot)$ is the joint cdf of $\Delta\boldsymbol{Y}(t;s)$, $F_{\Delta Y_i(t;s)}(\cdot),i=1,2,\ldots,N$ are the marginal cdfs of $\Delta\boldsymbol{Y}(t;s)$ and $\mathscr{C}_t(\cdot)$ is the copula function unrelated to the beginning time $s$ but related to the time interval $t$.

Five copulas with different association characteristics are considered in this study: Clayton copula, Frank copula and Gumbel copula are three one parameter (denoted as $\theta$) copulas from the Archimedean copulas class; normal copula and t copula belong to the elliptical copulas class. An Archimedean copula is defined as

$$\mathscr{C}(u_1,u_2,\ldots,u_N)=\varphi^{-1}(\varphi(u_1)+\varphi(u_2)+\cdots+\varphi(u_N))$$

where the generator $\varphi:[0,1]\rightarrow[0,\infty]$ is a continuous strictly decreasing function and $\varphi^{-1}(\cdot)$ is its inverse. The generators for Clayton copula, Frank copula and Gumbel copula are $u^{-\theta}-1$, $-\ln\left(\frac{e^{-\theta u}-1}{e^{-\theta}-1}\right)$ and $(-\ln u)^\theta$ respectively. The normal copula is defined as $\Phi_N(\Phi^{-1}(u_1),\Phi^{-1}(u_2),\cdots,\Phi^{-1}(u_N))$, where $\Phi_N(\cdot)$ is the joint cdf of a multivariate normal distribution with mean vector 0 and symmetric correlation matrix $\mathbf{R}=[r_{ij}]_{1\leq i,j\leq N}$ whose diagonal elements equal to 1. Similarly, the t copula is defined as $F_{N,\nu}(F_\nu^{-1}(u_1),F_\nu^{-1}(u_2),\cdots,F_\nu^{-1}(u_N))$, where $F_\nu(\cdot)$ is the cdf of the univariate Student's t distribution with $\nu$ degrees of freedom and $F_{N,\nu}(\cdot)$ is the joint cdf of a multivariate t distribution. For $N$ independent components, we can use the independence copula $\mathscr{I}(\cdot)$, which is defined as $\mathscr{I}(u_1,u_2,\ldots,u_N)=u_1u_2\cdots u_N$.

Kendall's tau ($\tau$), lower tail dependence ($\lambda_L$) and upper tail dependence ($\lambda_U$) are three commonly used measures of association for bivariate copulas [27]. These measures for the five copulas are summarized in Table 1, where $\mathbb{I}_{\{\cdot\}}$ is the indicator function. Kendall's tau measures the rank correlation, and the larger the value of $\tau$, the stronger the dependence. From Table 1, we can find that $\tau$ is a monotone increasing function of the parameter ($\theta$ or $r_{12}$) for the five copulas. Tail dependence summarizes the dependence in the tails of the bivariate distributions. We can find from Table 1 that the five copulas show different tail dependence: Frank copula has no tail dependence, Clayton copula has lower tail dependence, Gumbel copula has upper tail dependence, normal copula has both lower and upper tail dependence only when $r_{12} = 1$, and t copula has both lower and upper tail dependence. Therefore, the five copulas considered in this study are representative and can characterize various dependence scenarios.

After introducing the dependence structure using copulas, we are able to investigate when the multi-component system will fail. The failure time $T$ for a $K$-out-of-$N$: G system is defined as

$$T = \inf_t \left\{ \sum_{i=1}^{N} \mathbb{I}_{\{T_i \le t\}} \ge N - K + 1 \right\} \tag{2-13}$$

where $T_i,\ i = 1, 2, \dots, N$ is the FHT defined in Eq. (2-1). Let $F_T(t)$ denote the cdf of $T$, then we have

$$\begin{aligned} F_T(t) = P(T \le t) &= \sum_{k=N-K+1}^{N} P(\text{exactly } k \text{ components fail at time } t) \\ &= \sum_{k=N-K+1}^{N} \sum_{s \in S_k} P(\{T_i \le t, \forall i \in s\} \cap \{T_i > t, \forall i \notin s\}) \end{aligned} \tag{2-14}$$

where $S_k,\ k = N-K+1, \dots, N$ is the set of all possible combinations of $k$-component failure with $|S_k| = C_N^k$, and $s \in S_k$ is a subset of $\{1, 2, \dots, N\}$ with $|s| = k$, which contains the index of the $k$ failed components. For example, for a 2-out-of-3: G system, we have $S_2 = \{\{1,2\},\{1,3\},\{2,3\}\}$, $S_3 = \{\{1,2,3\}\}$.

For Gamma and IG degradation processes, due to their monotonicity, $T_i \le t$ is equivalent to $Y_i(t) \ge L_i$. Thus Eq. (2-14) can be revised as

$$\begin{aligned} F_T(t) &= \sum_{k=N-K+1}^{N} \sum_{s \in S_k} P(\{Y_i(t) \ge L_i, \forall i \in s\} \cap \{Y_i(t) < L_i, \forall i \notin s\}) \\ &= \sum_{k=N-K+1}^{N} \sum_{s \in S_k} \int_{ll_1(s)}^{ul_1(s)} \int_{ll_2(s)}^{ul_2(s)} \cdots \int_{ll_N(s)}^{ul_N(s)} f_{\Delta Y(t;0)}(y_1, y_2, \dots, y_N) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_N \end{aligned} \tag{2-15}$$

where $ll_i(s) = \begin{cases} L_i & i \in s \\ 0 & i \notin s \end{cases}$ and $ul_i(s) = \begin{cases} \infty & i \in s \\ L_i & i \notin s \end{cases}$ are the lower and upper limit of integral respectively for $i = 1, 2, \dots, N$, and $f_{\Delta Y(t;0)}(\cdot)$ is the joint pdf of $\Delta \mathbf{Y}(t; 0)$, which can be derived analogous to Eq. (2-10).

For Wiener degradation process, as it is non-monotonic, we can't construct equivalence between $T_i \le t$ and $Y_i(t) \ge L_i$. However, we know that $T_i, i = 1, 2, \dots, N$ is IG distributed for Wiener process with cdf $F_{T_i}(\cdot)$ given in Eq. (2-5). If the degradation processes of all components are independent, or equivalently, $\mathscr{C}_t(\cdot) = \mathscr{I}(\cdot)$, then $F_T(t)$ can be calculated by

$$F_T(t) = \sum_{k=N-K+1}^{N} \sum_{s \in S_k} \left( \prod_{i \in s} F_{T_i}(t) \prod_{i \notin s} (1 - F_{T_i}(t)) \right) \tag{2-16}$$

Note that this expression of $F_T(t)$ is also applicable to the case of independent Gamma or IG degradation processes. If $\mathscr{C}_t(\cdot) \ne \mathscr{I}(\cdot)$, $F_T(t)$ can be computed with the aid of simulation.

### 2.4. Imperfect maintenance

Imperfect maintenance is common in practice and the damage reduction of imperfect maintenance is often modeled by a random variable. For example, Wang et al. [24] used negative jumps in the degradation state to quantify the influence of imperfect maintenance, Zhao et al. [28] used an improvement factor subject to truncated normal distribution in the range $[0, 1]$ to reflect the reduced degradation level, and Cheng et al. [29] used a Beta distributed random variable to represent the deterioration level after PM. Motivated by them, we use a reduction in the degradation level to indicate imperfect maintenance.

Suppose that we perform PM on component $i$ at time $t_0$, then the degradation process after PM is

$$Y_i(t_0 + t) = Y_i(t_0^-) - PM_{i, Y_i(t_0^-)} + \Delta Y_i(t; t_0), t > 0 \tag{2-17}$$

where $Y_i(t_0^-)$ is the degradation level of component $i$ immediately before PM, $PM_{i, Y_i(t_0^-)}$ is a random variable representing the reduction in the degradation level, and $\Delta Y_i(t; t_0)$ is the degradation increment from time $t_0$ to $t_0 + t$ for $i = 1, 2, \dots, N$. We further assume that $PM_{1, Y_1(t_0^-)}, PM_{2, Y_2(t_0^-)}, \dots, PM_{N, Y_N(t_0^-)}$ are independent with each other, as mentioned in Assumption A7. The distribution of $PM_{i, Y_i(t_0^-)}$ should be determined based on historical data or domain knowledge. In this study, we assume that the cdf of $PM_{i, Y_i(t_0^-)}$ takes the form of

$$F_{PM_{i, Y_i(t_0^-)}}(y) = \begin{cases} (1 - p_i) F_{\mathscr{B}e}\left( \dfrac{y}{Y_i(t_0^-)}; a_i, b_i \right) & 0 \le y < Y_i(t_0^-) \\ 1 & y = Y_i(t_0^-) \end{cases} \tag{2-18}$$

where $p_i \in [0, 1]$ is the probability that component $i$ after PM is as-good-as-new, and $F_{\mathscr{B}e}(y; a_i, b_i) = \int_0^y \frac{1}{\mathscr{B}(a_i, b_i)} x^{a_i - 1} (1 - x)^{b_i - 1} \mathrm{d}x$ is the cdf of Beta distribution $\mathscr{B}e(a_i,\ b_i)$, $a_i > 0$, $b_i > 0$, with $\mathscr{B}(a_i, b_i) = \int_0^1 x^{a_i - 1} (1 - x)^{b_i - 1} \mathrm{d}x$ being the Beta function. The above formulation is very flexible. We can adjust $a_i, b_i$ (the two parameters of Beta distribution) and $p_i$ to represent imperfect maintenance with different characteristics. Keeping $a_i, b_i$ unchanged, $1 - p_i$ can be treated as an indicator of the degree of imperfect maintenance. For the case of perfect PM, we can simply set $p_i = 1$.

## 3. Optimization method

### 3.1. The expected long-run discounted cost

In this study, the time horizon is considered to be infinite. To handle the infinite time horizon and take the time value of cost into consideration, the cost incurred at time $t$ is discounted with $e^{-rt}$, where $r$ is the discount factor. Our optimization objective is to minimize the expected long-run discounted cost for a brand new multi-component system. Based on the aforementioned maintenance settings of the multi-component system in Section 2.1, the costs incurred include two parts: inspection cost and maintenance cost.

Inspection cost $c_i$ occurs every $\delta$ time units. Thus, the expected long-run discounted inspection cost is

$$I_\delta = \sum_{j=0}^{\infty} c_i e^{-rj\delta} = \frac{c_i}{1 - e^{-r\delta}} \tag{3-1}$$

The expected long-run discounted maintenance cost is related to the initial degradation levels of the $N$ components. Let $y_1, y_2 \dots, y_N$ denote the current degradation levels of component $1, 2, \dots, N$ upon inspection, and $\mathbf{y} = [y_1, y_2 \dots, y_N]$ be the corresponding vector. We denote by $V_\delta(\mathbf{y}) = V_\delta(y_1, y_2, \dots, y_N)$ the expected long-run discounted maintenance cost when the inspection interval is $\delta$ and the current degradation levels are $\mathbf{y}$. In complex scenarios, the explicit form of $V_\delta(\mathbf{y})$ is hard to derive. However, due to the Markov property of the system, we use MDP to solve for $V_\delta(\mathbf{y})$, which will be presented in the next section.

Our objective is to find the optimal inspection interval $\delta^*$ and the optimal CBM policy to minimize the total expected long-run discounted cost for a brand new system, that is, $I_\delta + V_\delta(0, 0, \dots, 0)$. It is worthwhile to mention that the model is also applicable to systems with other initial conditions.

### 3.2. MDP model for minimizing the expected long-run discounted maintenance cost

In this section, we utilize the MDP framework to find $V_\delta(\mathbf{y})$ under a given inspection interval $\delta$. The basic elements of the MDP model are given below.

(1) **State**

The continuous state $\mathbf{y} = [y_1, y_2 \ldots, y_N]$ represents the degradation levels of $N$ components upon inspection, and the state space $S$ is defined as

$$S = \{\mathbf{y} \in \mathbb{R}^N | y_i \geq 0, i = 1, 2, \ldots, N\} \tag{3-2}$$

Due to the heterogeneity among components, we normalize $y_i, i = 1, 2, \ldots, N$ by the failure threshold $L_i$ so that we can compare the failure extent of the $N$ components and then find the top $n \in \{1, 2, \ldots, N\}$ components with the worst conditions, which is easier to implement in practice. The failure threshold-normalized degradation level $\widetilde{y}_i$ is defined as

$$\widetilde{y}_i = \frac{y_i}{L_i}, \ i = 1, 2, \ldots, N \tag{3-3}$$

We can see that the larger the value of $\widetilde{y}_i$, the higher the failure extent. Let $\widetilde{y}_{(1)} \leq \widetilde{y}_{(2)} \leq \ldots \leq \widetilde{y}_{(N)}$ denote the order statistics of $\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_N$ and $r_1, r_2, \ldots, r_N$ be the rank of $\widetilde{y}_{(1)}, \widetilde{y}_{(2)}, \ldots, \widetilde{y}_{(N)}$ in $\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_N$. Then we have $\widetilde{y}_{r_i} = \widetilde{y}_{(i)}, \ i = 1, 2, \ldots, N$, thus $\widetilde{y}_{r_1} \leq \widetilde{y}_{r_2} \leq \ldots \leq \widetilde{y}_{r_N}$ and the top $n \in \{1, 2, \ldots, N\}$ components with the worst conditions are component $r_{N-n+1}, r_{N-n+2}, \ldots, r_N$.

For the $K$-out-of-$N$: G system, it will fail when less than $K$ components are working in normal conditions, or mathematically, when $\widetilde{y}_{(K)} \geq 1$. The set of all system failure states is defined as

$$S^F = \left\{\mathbf{y} \in S | \widetilde{y}_{(K)} \geq 1\right\} \tag{3-4}$$

(2) **Action**

Upon each inspection, if the system is found failed (i.e., $\mathbf{y} \in S^F$), CM is performed. Otherwise, there are totally $N+1$ possible actions: to do nothing (NULL) or to perform PM on $n \in \{1, 2, \ldots, N\}$ components with the worst conditions ($\text{PM}_n$). Thus, the action space in state $\mathbf{y}$ is

$$A(\mathbf{y}) = \begin{cases} \{\text{CM}\} & \mathbf{y} \in S^F \\ \{\text{NULL}, \ \text{PM}_1, \text{PM}_2, \ldots, \text{PM}_N\} & \mathbf{y} \notin S^F \end{cases} \tag{3-5}$$

(3) **CBM policy**

We adopt a deterministic policy $\pi_\delta : S \rightarrow \{\text{CM}, \text{NULL}, \ \text{PM}_1, \text{PM}_2, \ldots, \text{PM}_N\}$. If the state upon inspection is $\mathbf{y}$, action $a = \pi_\delta(\mathbf{y})$ is taken. In practice, when $\mathbf{y} \in S^F$, $\pi_\delta(\mathbf{y}) = \text{CM}$. Our main focus is to solve the optimal CBM policy when $\mathbf{y} \notin S^F$.

(4) **State transition function**

Let $\mathbf{y} = [y_1, y_2 \ldots, y_N]$ be the state upon the current inspection, $\mathbf{y}^A = [y_1^A, y_2^A, \ldots, y_N^A]$ be the state immediately after taking action $a \in A(\mathbf{y})$, and $\mathbf{y}' = [y_1', y_2', \ldots, y_N']$ be the state upon the next inspection. Denote $\mathbf{Y} = [Y_1, Y_2 \ldots, Y_N]$, $\mathbf{Y}^A = [Y_1^A, Y_2^A, \ldots, Y_N^A]$ and $\mathbf{Y}' = [Y_1', Y_2', \ldots, Y_N']$ as the corresponding random vectors of $\mathbf{y}$, $\mathbf{y}^A$ and $\mathbf{y}'$, and denote $A$ as the random variable of action $a$. Although the value of $A$ should be real number, for the ease of readability, we specify that $A = CM, A = \text{NULL}, A = \text{PM}_i, i = 1, 2, \ldots, N$ represent $A = -1, A = 0, A = i, i = 1, 2, \ldots, N$ in the following context. The state transition between two successive inspection epochs can be represented by the Bayesian network shown in Fig. 1, where each node represents a random vector (variable) and each directed arc indicates probabilistic relationships [30]. As $\mathbf{Y}'$ only depends on $\mathbf{Y}^A$, there is no directed arc from $\mathbf{Y}, A$ to $\mathbf{Y}'$.

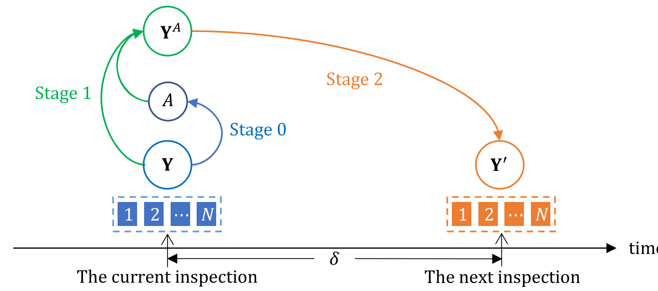


**Fig. 1.** Bayesian network representation of the state transition between two successive inspection epochs.

The state transition can be divided to three stages, as illustrated below.

**Stage 0**: Given the state $\mathbf{Y} = \mathbf{y}$ upon the current inspection, action $A = a$ is chosen based on the deterministic policy $a = \pi_\delta(\mathbf{y})$.
**Stage 1**: Given the state $\mathbf{Y} = \mathbf{y}$ and action $A = a$ upon the current inspection, the state will transition to $\mathbf{Y}^A = \mathbf{y}^A$ immediately with transition function $P_1(\mathbf{Y}^A = \mathbf{y}^A | \mathbf{Y} = \mathbf{y}, A = a)$.
**Stage 2**: After a period of length $\delta$, the state will transition from $\mathbf{Y}^A = \mathbf{y}^A$ to $\mathbf{Y}' = \mathbf{y}'$ upon the next inspection, with transition function $P_2(\mathbf{Y}' = \mathbf{y}' | \mathbf{Y}^A = \mathbf{y}^A)$.

Denote $P_o(\mathbf{Y}' = \mathbf{y}' | \mathbf{Y} = \mathbf{y}, A = a)$ as the overall transition function from state $\mathbf{Y} = \mathbf{y}$ with action $A = a$ to state $\mathbf{Y}' = \mathbf{y}'$. For the sake of brevity, we denote $P(\mathbf{y}' | \mathbf{y}, \ a) = P_o(\mathbf{Y}' = \mathbf{y}' | \mathbf{Y} = \mathbf{y}, \ A = a)$, $P_\delta(\mathbf{y}' | \mathbf{y}^A) = P_2(\mathbf{Y}' = \mathbf{y}' | \mathbf{Y}^A = \mathbf{y}^A)$ and $P_A(\mathbf{y}^A | \mathbf{y}, a) = P_1(\mathbf{Y}^A = \mathbf{y}^A | \mathbf{Y} = \mathbf{y}, A = a)$. The above three transition functions are derived as follows.

- Derivation of $P_A(\mathbf{y}^A | \mathbf{y}, a)$

For the ease of derivation, the following three cases are considered.

**Case 1**: If the system fails upon the current inspection ($\mathbf{Y} = \mathbf{y} \in S^F$), CM needs to be performed immediately ($A = \text{CM}$) and the system becomes as-good-as-new ($\mathbf{Y}^A = 0$). The state space of $\mathbf{Y}^A | \mathbf{Y} = \mathbf{y} \in S^F, A = \text{CM}$ is $S^A \mathbf{y}, \text{CM} = \{0\}$. Therefore, this is a deterministic transition and $P_A(\mathbf{y}^A | \mathbf{y}, \text{CM})$ specifies a probability mass function (pmf) with only one possible value, which can be expressed as

$$P_A(\mathbf{y}^A | \mathbf{y}, \text{CM}) = \begin{cases} 1 & \mathbf{y}^A = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3-6}$$

**Case 2**: If the system is normal upon the current inspection ($\mathbf{Y} = \mathbf{y} \notin S^F$) and we decide to do nothing ($A = \text{NULL}$), the state keeps unchanged ($\mathbf{Y}^A = \mathbf{y}$). The state space of $\mathbf{Y}^A | \mathbf{Y} = \mathbf{y} \notin S^F, A = \text{NULL}$ is $S^A \mathbf{y}, \ \text{NULL} = \{\mathbf{y}\}$. Therefore, this is also a deterministic transition and $P_A(\mathbf{y}^A | \mathbf{y}, \text{NULL})$ specifies a pmf with only one possible value, which can be expressed as

$$P_A(\mathbf{y}^A | \mathbf{y}, \text{NULL}) = \begin{cases} 1 & \mathbf{y}^A = \mathbf{y} \\ 0 & \text{otherwise} \end{cases} \tag{3-7}$$

**Case 3**: If the system is normal upon the current inspection ($\mathbf{Y} = \mathbf{y} \notin S^F$) and we decide to perform PM on the top $n$ worst components

($A = \mathrm{PM}_n, n = 1,2,\ldots,N$), components $r_{N-n+1},\ldots,r_N$ are maintained. Under the assumption that PM is imperfect (Assumption A7), the degradation level of component $i$ immediately after PM is $Y_i^A = y_i - PM_{i,y_i}$ instead of 0 for $i = r_{N-n+1},\ldots,r_N$ (See Section 2.4 for details). As we do nothing for components $r_1,\ldots,r_{N-n}$, $Y_{r_1}^A,\ldots,Y_{r_{N-n}}^A$ are deterministic with $Y_i^A = y_i$, $i = r_1,\ldots,r_{N-n}$. Thus, the state $\mathbf{Y}^A = [Y_1^A, Y_2^A,\ldots,Y_N^A]$ after $\mathrm{PM}_n$ is

$$Y_i^A = \begin{cases} y_i & i = r_1, r_2, \ldots, r_{N-n} \\ y_i - PM_{i,y_i} & i = r_{N-n+1}, r_{N-n+2}, \ldots, r_N \end{cases} \tag{3-8}$$

Then the state space of $\mathbf{Y}^A|\mathbf{Y} = \mathbf{y} \notin S^F, A = \mathrm{PM}_n$ is

$$S^A\mathbf{y}, \mathrm{PM}_n = \{\mathbf{y}^A \in S | y_i^A = y_i, \forall i = r_1,\ldots,r_{N-n} \text{ and } 0 \le y_i^A \le y_i, \forall i = r_{N-n+1},\ldots,r_N\} \tag{3-9}$$

which is continuous for $r_{N-n+1},\ldots,r_N$ and is deterministic for the remaining $N-n$ components. Therefore, the transition function $P_A(\mathbf{y}^A|\mathbf{y},\mathrm{PM}_n)$ specifies a joint pdf of $Y_{r_{N-n+1}}^A,\ldots,Y_{r_N}^A$ and $Y_i^A$ is deterministic with value $y_i$ for $i = r_1,\ldots,r_{N-n}$ [31]. As we have assumed that $PM_{i,y_i}$ are independent (Assumption A7), $y_i - PM_{i,y_i}, i = r_{N-n+1},\ldots,r_N$ are independent as well. Thus, we have

$$P_A(\mathbf{y}^A|\mathbf{y},\mathrm{PM}_n) = \begin{cases} \prod\limits_{i=r_{N-n+1},\ldots,r_N} f_{PM_{i,y_i}}(y_i - y_i^A) & \mathbf{y}^A \in S^A\mathbf{y},\mathrm{PM}_n \\ 0 & \text{otherwise} \end{cases} \tag{3-10}$$

where $f_{PM_{i,y_i}}(\cdot), i = r_{N-n+1},\ldots,r_N$ is the pdf of $PM_{i,y_i}$, which can be derived from the cdf in Eq. (2-18).

- **Derivation of $P_\delta(\mathbf{y}'|\mathbf{y}^A)$**

As the state space of $\mathbf{Y}'$ are continuous, the transition function $P_\delta(\mathbf{y}'|\mathbf{y}^A)$ specifies a joint pdf with $N$ dimensions [31]. To compute $P_\delta(\mathbf{y}'|\mathbf{y}^A)$, due to the independent increments, the degradation processes from $\mathbf{y}^A$ to $\mathbf{y}'$ with inspection interval $\delta$ can be treated as the degradation processes from 0 to $\Delta\mathbf{y} = \mathbf{y}' - \mathbf{y}^A = [y_1' - y_1^A, y_2' - y_2^A, \ldots, y_N' - y_N^A] = [\Delta y_1, \Delta y_2, \ldots, \Delta y_N]$ with adjusted failure thresholds:

$$L_i' = L_i - y_i^A, i = 1,2,\ldots,N \tag{3-11}$$

Let $\Delta Y_i, i = 1,2,\ldots,N$ be the random variable representing the degradation increment $\Delta y_i$ of component $i$, and let $\Delta\mathbf{Y} = [\Delta Y_1, \Delta Y_2, \ldots, \Delta Y_N]$ be the corresponding random vector. Then, the derivation of $P_\delta(\mathbf{y}'|\mathbf{y}^A)$ is similar to the derivation of the distribution of the system failure time in Section 2.3.

For the monotonic Gamma process and IG process, $P_\delta(\mathbf{y}'|\mathbf{y}^A)$ can be computed based on the degradation increments dependence characterized in Eq. (2-12) as

$$P_\delta(\mathbf{y}'|\mathbf{y}^A) = \frac{\partial \mathscr{C}_\delta(F_{\Delta Y_1}(\Delta y_1),\ldots,F_{\Delta Y_N}(\Delta y_N))}{\partial F_{\Delta Y_1}\partial F_{\Delta Y_2}\cdots\partial F_{\Delta Y_N}} f_{\Delta Y_1}(\Delta y_1) f_{\Delta Y_2}(\Delta y_1)\cdots f_{\Delta Y_N}(\Delta y_1) \tag{3-12}$$

where $F_{\Delta Y_i}(\cdot), f_{\Delta Y_i}(\cdot), i = 1,2,\ldots,N$ are the marginal cdfs and pdfs of $\Delta\mathbf{Y}$, and $C_\delta(\cdot,\cdot,\ldots,\cdot)$ is the copula function under interval $\delta$. In particular, $F_{\Delta Y_i}(\cdot) = F_{\mathscr{Ga}}(\cdot;\alpha_i\delta,\beta_i)$ for Gamma process and $F_{\Delta Y_i}(\cdot) = F_{\mathscr{IG}}(\cdot;\nu_i\delta,\lambda_i\delta^2)$ for IG process. For Wiener process, we can't compute $P_\delta(\mathbf{y}'|\mathbf{y}^A)$ by Eq. (3-12) as it is not monotonic. But with the aid of simulation, $P_\delta(\mathbf{y}'|\mathbf{y}^A)$ can be computed for Wiener process.

- **Derivation of $P(\mathbf{y}'|\mathbf{y},a)$**

As the state space of $\mathbf{Y}'$ are continuous, the overall transition function $P(\mathbf{y}'|\mathbf{y},a)$ specifies a joint pdf with $N$ dimensions [31]. The same three cases as in the derivation of $P_A(\mathbf{y}^A|\mathbf{y},a)$ are considered.

**Case 1:** If $\mathbf{Y} = \mathbf{y} \in S^F$, then $A = \mathrm{CM}$ and the state transitions to $\mathbf{Y}^A = 0$ deterministically, and then transitions to $\mathbf{Y}' = \mathbf{y}'$ upon the next inspection. The overall transition is equivalent to the transition from $\mathbf{Y}^A = 0$ to $\mathbf{Y}' = \mathbf{y}'$ in stage 2. Thus, we have $P(\mathbf{y}'|\mathbf{y},\mathrm{CM}) = P_\delta(\mathbf{y}'|0)$ for $\mathbf{y} \in S^F$.
**Case 2:** If $\mathbf{Y} = \mathbf{y} \notin S^F$ and $A = \mathrm{NULL}$, then the state transitions to $\mathbf{Y}^A = \mathbf{y}$ deterministically, and then transitions to $\mathbf{Y}' = \mathbf{y}'$ upon the next inspection. The overall transition is equivalent to the transition from $\mathbf{Y}^A = \mathbf{y}$ to $\mathbf{Y}' = \mathbf{y}'$ in stage 2. Thus, we have $P(\mathbf{y}'|\mathbf{y},\mathrm{NULL}) = P_\delta(\mathbf{y}'|\mathbf{y})$ for $\mathbf{y} \notin S^F$.
**Case 3:** If $\mathbf{Y} = \mathbf{y} \notin S^F$ and $A = \mathrm{PM}_n$, then the state transitions to $\mathbf{Y}^A = \mathbf{y}^A \in S^A\mathbf{y},\mathrm{PM}_n$ with pdf $P_A(\mathbf{y}^A|\mathbf{y},\mathrm{PM}_n)$, and then transitions to $\mathbf{Y}' = \mathbf{y}'$ upon the next inspection. According the Bayesian network shown in Fig. 1, the joint conditional pdf of $\mathbf{Y}^A, \mathbf{Y}'$ given $\mathbf{Y} = \mathbf{y} \notin S^F$ and $A = \mathrm{PM}_n$ can be factorized as the product of the two conditional pdfs, that is, $P_A(\mathbf{y}^A|\mathbf{y},\mathrm{PM}_n)P_\delta(\mathbf{y}'|\mathbf{y}^A)$, which is $n+N$-dimensional. By marginalizing the $n$ continuous components of $\mathbf{Y}^A$, that is, $Y_{r_{N-n+1}}^A,\ldots,Y_{r_N}^A$, we obtain the overall transition function as $P(\mathbf{y}'|\mathbf{y},\mathrm{PM}_n) = \int_0^{y_{r_{N-n+1}}}\cdots\int_0^{y_{r_N}} P_A(\mathbf{y}^A|\mathbf{y},\mathrm{PM}_n)P_\delta(\mathbf{y}'|\mathbf{y}^A)\mathrm{d}y'_{r_{N-n+1}}\cdots\mathrm{d}y'_{r_N}$ with $y_i^A = y_i$ for $i = r_1,\ldots,r_{N-n}$.

Above all, the transition function from state $\mathbf{Y} = \mathbf{y}$ with action $A = a$ to state $\mathbf{Y}' = \mathbf{y}'$ is

$$P(\mathbf{y}'|\mathbf{y},a) = \begin{cases} P_\delta(\mathbf{y}'|0) & \mathbf{y} \in S^F, a = \mathrm{CM} \\ P_\delta(\mathbf{y}'|\mathbf{y}) & \mathbf{y} \notin S^F, a = \mathrm{NULL} \\ \int_0^{y_{r_{N-n+1}}}\cdots\int_0^{y_{r_N}} P_A(\mathbf{y}^A|\mathbf{y},\mathrm{PM}_n)P_\delta(\mathbf{y}'|\mathbf{y}^A)\mathrm{d}y^A_{r_{N-n+1}}\cdots\mathrm{d}y^A_{r_N} & \mathbf{y} \notin S^F, a = \mathrm{PM}_n, n = 1,2,\ldots,N \end{cases} \tag{3-13}$$

Thus, it can be verified that

$$\mathbb{E}[g(\mathbf{Y}')|\mathbf{Y} = \mathbf{y}, A = a] = \mathbb{E}\left[\mathbb{E}\left[g(\mathbf{Y}')|\mathbf{Y}^A\right]|\mathbf{Y} = \mathbf{y}, A = a\right] \tag{3-14}$$

where $g(\mathbf{Y}')$ is a function of the random vector $\mathbf{Y}'$.

(5)**Reward**

The expected cost of transition from state $\mathbf{Y} = \mathbf{y} \in S$ (the state upon the current inspection), taking action $A = a \in A(\mathbf{y})$ to $\mathbf{Y}^A = \mathbf{y}^A \in S$ (the state immediately after taking action), and finally to $\mathbf{Y}' = \mathbf{y}' \in S$ (the state upon the next inspection) is denoted as $C(\mathbf{y},a,\mathbf{y}^A,\mathbf{y}')$, which comprises two parts: (1) maintenance cost when performing CM or PM, and (2) expected downtime cost between two inspection epochs.

For the maintenance cost, when $\mathbf{Y} = \mathbf{y} \in S^F$, the system fails and we perform CM ($a$ = CM) with maintenance cost $c_c$ and $\mathbf{Y}^A = 0$. When $\mathbf{Y} = \mathbf{y} \notin S^F$ and $A \in \{\text{NULL}, \text{PM}_1, \text{PM}_2, \ldots, \text{PM}_N\}$: If we do nothing ($A$ = NULL), no maintenance cost is incurred and $\mathbf{Y}^A = \mathbf{y}$; If we perform PM on $n = 1, 2, \ldots, N$ components with the worst conditions ($A = \text{PM}_n$), the maintenance cost is $c_s + nc_p$.

For the downtime cost, it begins to accumulate with cost $c_d$ per time unit immediately after the moment when the system breaks down. Let $D_\delta(\mathbf{y}^A, \mathbf{y}')$ denote the expected downtime cost between two successive inspection epochs if the state after taking action is $\mathbf{Y}^A = \mathbf{y}^A$ and the state upon the next inspection is $\mathbf{Y}' = \mathbf{y}'$. Let $D_\delta(\mathbf{y}^A)$ denotes the expected downtime cost between two successive inspection epochs if the state after taking action is $\mathbf{Y}^A = \mathbf{y}^A$. Then the following relationship holds between $D_\delta(\mathbf{y}^A, \mathbf{y}')$ and $D_\delta(\mathbf{y}^A)$:

$$D_\delta(\mathbf{y}^A) = \mathbb{E}\left[D_\delta(\mathbf{y}^A, \mathbf{y}')|\mathbf{Y}^A = \mathbf{y}^A\right] \tag{3-15}$$

In order to compute $D_\delta(\mathbf{y}^A)$, we need to find the distribution of system failure time, which has been discussed in Section 2.3. Let $T_{\mathbf{y}^A}$ denote the system failure time with initial state $\mathbf{Y}^A = \mathbf{y}^A$, and let $f_{T_{\mathbf{y}^A}}(t)$, $F_{T_{\mathbf{y}^A}}(t)$ be the pdf and cdf of $T_{\mathbf{y}^A}$, respectively. Due to the independent increments, we can regard $T_{\mathbf{y}^A}$ as the failure time $T$ of a system beginning in state 0 with adjusted failure thresholds $L_i' = L_i - y_i^A, i = 1, 2, \ldots, N$. If the system fails at time $T_{\mathbf{y}^A} = t \in [0, \delta]$, the discounted downtime cost $DC(t)$ is

$$DC(t) = \int_t^\delta c_d e^{-r\tau} \mathrm{d}\tau = \frac{c_d(e^{-rt} - e^{-r\delta})}{r}, 0 \le t \le \delta \tag{3-16}$$

and thus $D_\delta(\mathbf{y}^A)$ is derived as

$$D_\delta(\mathbf{y}^A) = \int_0^\delta DC(t) f_{T_{\mathbf{y}^A}}(t)\mathrm{d}t = \int_0^\delta \frac{c_d(e^{-rt} - e^{-r\delta})}{r} f_{T_{\mathbf{y}^A}}(t)\mathrm{d}t \tag{3-17}$$

Above all, the expected cost is

$$C(\mathbf{y}, a, \mathbf{y}^A, \mathbf{y}') = \begin{cases} c_c + D_\delta(0, \mathbf{y}') & \mathbf{y} \in S^F, a = \text{CM} \\ D_\delta(\mathbf{y}, \mathbf{y}') & \mathbf{y} \notin S^F, a = \text{NULL} \\ c_s + nc_p + D_\delta(\mathbf{y}^A, \mathbf{y}') & \mathbf{y} \notin S^F, a = \text{PM}_n, n = 1, 2, \ldots, N \end{cases} \tag{3-18}$$

Under the MDP framework, the state value function is the expected long-run discounted maintenance cost $V_\delta(\mathbf{y})$ defined in Section 3.1. The state-action value function $Q_\delta(\mathbf{y}, a), \mathbf{y} \in S, a \in A(\mathbf{y})$ is defined as

$$\begin{aligned} Q_\delta(\mathbf{y}, a) &= \mathbb{E}\left[C(\mathbf{y}, a, \mathbf{Y}^A, \mathbf{Y}') + e^{-r\delta} V_\delta(\mathbf{Y}')|\mathbf{Y} = \mathbf{y}, A = a\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[C(\mathbf{y}, a, \mathbf{Y}^A, \mathbf{Y}') + e^{-r\delta} V_\delta(\mathbf{Y}')|\mathbf{Y}^A\right]|\mathbf{Y} = \mathbf{y}, A = a\right] \end{aligned} \tag{3-19}$$

where we use the relationship given in Eq. (3-14). Substituting Eq. (3-18) into Eq. (3-19) and utilizing the relationship that $D_\delta(\mathbf{y}^A) = \mathbb{E}[D_\delta(\mathbf{y}^A, \mathbf{y}')|\mathbf{Y}^A = \mathbf{y}^A]$ given in Eq. (3-15), we obtain

$$Q_\delta(\mathbf{y}, a) = \begin{cases} c_c + D_\delta(0) + e^{-r\delta}\mathbb{E}\left[V_\delta(\mathbf{Y}')|\mathbf{Y}^A = 0\right] & \mathbf{y} \in S^F, a = \text{CM} \\ D_\delta(\mathbf{y}) + e^{-r\delta}\mathbb{E}\left[V_\delta(\mathbf{Y}')|\mathbf{Y}^A = \mathbf{y}\right] & \mathbf{y} \notin S^F, a = \text{NULL} \\ c_s + nc_p + \mathbb{E}\left[D_\delta(\mathbf{Y}^A) + e^{-r\delta}\mathbb{E}\left[V_\delta(\mathbf{Y}')|\mathbf{Y}^A\right]|\mathbf{Y} = \mathbf{y}, A = a\right] & \mathbf{y} \notin S^F, a = \text{PM}_n, \\ & n = 1, 2, \ldots, N \end{cases} \tag{3-20}$$

Therefore, The Bellman equation of the state value function $V_\delta(\mathbf{y})$ is

$$V_\delta(\mathbf{y}) = \min_{a \in A(\mathbf{y})} Q_\delta(\mathbf{y}, a) = \begin{cases} Q_\delta(\mathbf{y}, \text{CM}) & \mathbf{y} \in S^F \\ \min\{Q_\delta(\mathbf{y}, \text{NULL}), Q_\delta(\mathbf{y}, \text{PM}_1), \ldots, Q_\delta(\mathbf{y}, \text{PM}_N)\} & \text{otherwise} \end{cases} \tag{3-21}$$

### 3.3. Solving the MDP problem by discretizing the state space

In order to solve the MDP problem, we discretize the continuous state space $S$ and then implement the value iteration algorithm with Monte Carlo simulation to find the optimal inspection interval and CBM policy. Without loss of generality, we assume that the degradation process of component $i$ will terminate once reaching its failure threshold $L_i$, thus we have $0 \le y_i \le L_i, i = 1, 2, \ldots, N$. Let $S_i = \{y_i | 0 \le y_i \le L_i\}$, $i = 1, 2, \ldots, N$ denote the degradation state space for component $i$. We discretize $S_i$ into $d_i$ equidistant intervals, where $d_i$ is a positive integer. Let $\Delta_i = L_i / d_i$ denote the discretization level, then the state space after discretization for component $i$ is $S_i^\Delta = \{0, \Delta_i, 2\Delta_i, \ldots, L_i\}, i = 1, 2, \ldots, N$, where the superscript $\Delta$ hereafter is used to indicate the notations after discretization. Thus, the state space for the system after discretization is

$$S^\Delta = \left\{\mathbf{y}^\Delta = \left[y_1^\Delta, y_2^\Delta \ldots, y_N^\Delta\right] | y_i^\Delta \in S_i^\Delta, i = 1, 2, \ldots, N\right\} \tag{3-22}$$

and the set of system failure states after discretization is

$$S^{F\Delta} = \left\{\mathbf{y}^\Delta \in S^\Delta | \widetilde{y}_{(K)}^\Delta \ge 1\right\} \tag{3-23}$$

After discretizing the state space, we can derive the discretized form of the transition functions accordingly. The discretized transition function from state $\mathbf{y}^\Delta \in S^\Delta$ to state $\mathbf{y}^{A\Delta} \in S^\Delta$ immediately after taking action $a \in A(\mathbf{y}^\Delta)$ is computed by

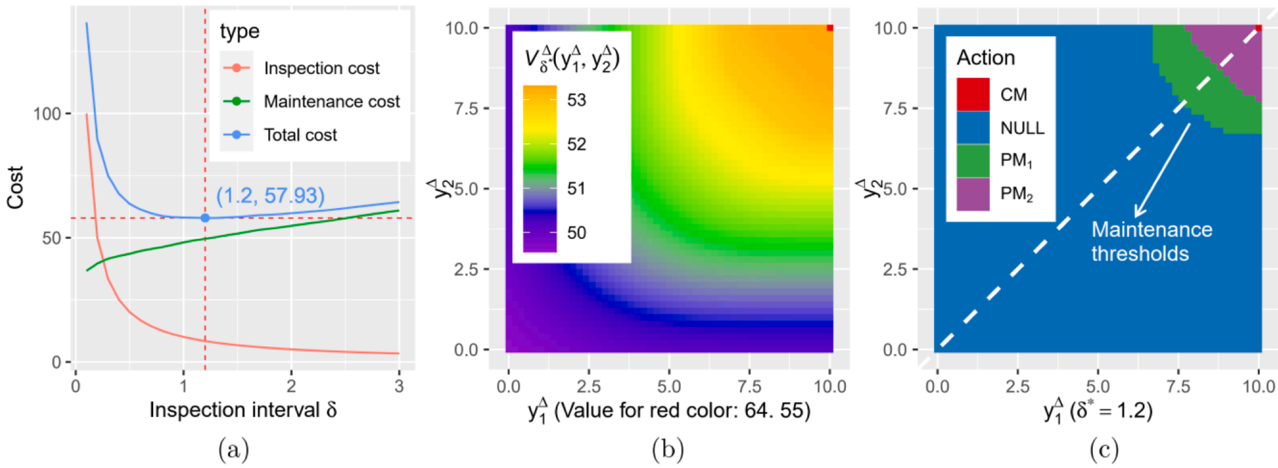


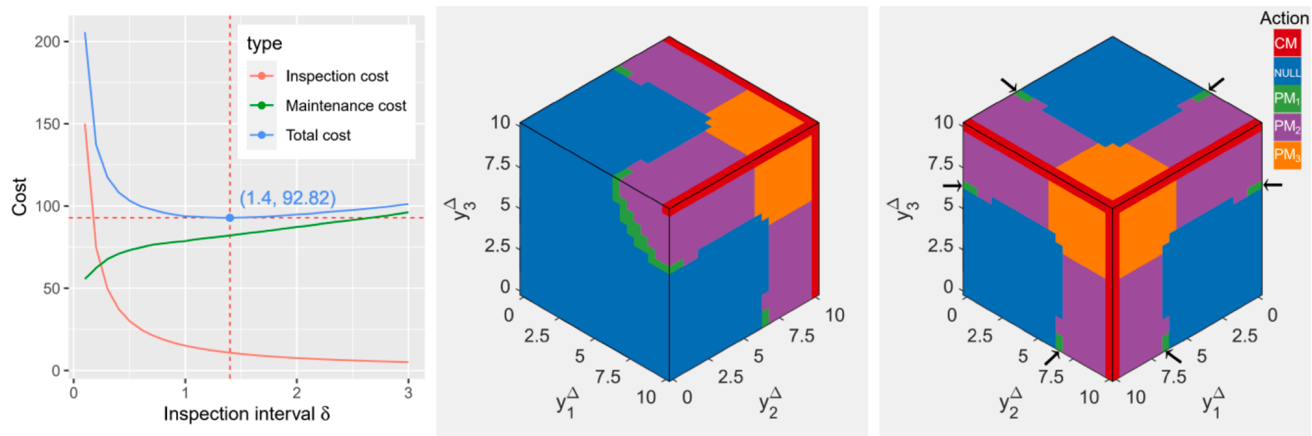

**Fig. 2.** (a) Line charts between the cost and the inspection interval $\delta$. (b) The heat map of the value function under $\delta^* = 1.2$. (c) The policy map of the CBM policy under $\delta^* = 1.2$.

$$P_A^\Delta(\mathbf{y}^{A\Delta}|\mathbf{y}^\Delta,a) = \begin{cases} 1 & \mathbf{y}^\Delta \in S^{F\Delta}, a=\text{CM},\ \mathbf{y}^{A\Delta}=0 \text{ or} \\ & \mathbf{y}^\Delta \notin S^{F\Delta}, a=\text{NULL},\ \mathbf{y}^{A\Delta}=\mathbf{y}^\Delta \\ \prod\limits_{i=r_{N-n+1},\ldots,r_N}\left[F_{PM_{i,y_i^\Delta}}\left(y_i^\Delta - y_i^{A\Delta}+\frac{\Delta_i}{2}\right) - F_{PM_{i,y_i^\Delta}}\left(y_i^\Delta - y_i^{A\Delta}-\frac{\Delta_i}{2}\right)\right] & \mathbf{y}^\Delta\notin S^{F\Delta}, a=\text{PM}_n, \mathbf{y}^{A\Delta}\in S^{A\Delta}\mathbf{y}^\Delta,\text{PM}_n, n=1,2,\ldots,N \\ 1 & \text{otherwise}\end{cases} \tag{3-24}$$

where $F_{PM_{i,y_i^\Delta}}(\cdot), i=r_{N-n+1},\ldots,r_N$ is the cdf of $PM_{i,y_i^\Delta}$ given by Eq. (2-18), and $S^{A\Delta}\mathbf{y}^\Delta, \text{PM}_n$ is the set of all possible discretized states after $\text{PM}_n$ in state $\mathbf{y}^\Delta$, which is analogous to Eq. (3-9). Eq. (3-24) can be interpreted as follows. When $\mathbf{y}^\Delta \in S^{F\Delta}, a = \text{CM}$ or $\mathbf{y}^\Delta \notin S^{F\Delta}, a = \text{NULL}$, $P_A^\Delta(\mathbf{y}^{A\Delta}|\mathbf{y}^\Delta, \text{CM})$ and $P_A^\Delta(\mathbf{y}^{A\Delta}|\mathbf{y}^\Delta, \text{NULL})$ have the same expressions as Eq. (3-6) and Eq. (3-7), respectively. When $\mathbf{y}^\Delta \notin S^{F\Delta}, a = \text{PM}_n, n = 1,2,\ldots,N$, since $PM_{i,y_i^\Delta}, i = r_{N-n+1},\ldots,r_N$ are assumed to be independent, we can first compute the probability that the state of component $i$ after $\text{PM}_n$ is $y_i^{A\Delta}$, denoted as $P_A^\Delta(y_i^{A\Delta}|y_i^\Delta, \text{PM})$, and then multiply them together to get $P_A^\Delta(\mathbf{y}^{A\Delta}|\mathbf{y}^\Delta, \text{PM}_n)$. Motivated by the idea of continuity correction, we compute $P_A^\Delta(y_i^{A\Delta}|y_i^\Delta, \text{PM})$ by assigning the probability in the small interval $\left[y_i^{A\Delta} - \frac{\Delta_i}{2}, y_i^{A\Delta} + \frac{\Delta_i}{2}\right]$ to the discretized point $y_i^{A\Delta}$: $P_A^\Delta(y_i^{A\Delta}|y_i^\Delta, \text{PM}) = F_{PM_{i,y_i^\Delta}}\left(y_i^\Delta - y_i^{A\Delta} + \frac{\Delta_i}{2}\right) - F_{PM_{i,y_i^\Delta}}\left(y_i^\Delta - y_i^{A\Delta} - \frac{\Delta_i}{2}\right)$.

The discretized transition function from state $\mathbf{y}^{A\Delta} \in S^\Delta$ to state $\mathbf{y}'^\Delta \in S^\Delta$ upon the next inspection, denoted as $P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta})$, is computed by

$$P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta}) = \int_{\max\left\{0, y_1'^\Delta - \frac{\Delta_1}{2}\right\}}^{\min\left\{L_1, y_1'^\Delta + \frac{\Delta_1}{2}\right\}} \int_{\max\left\{0, y_2'^\Delta - \frac{\Delta_2}{2}\right\}}^{\min\left\{L_2, y_2'^\Delta + \frac{\Delta_2}{2}\right\}} \cdots \int_{\max\left\{0, y_N'^\Delta - \frac{\Delta_N}{2}\right\}}^{\min\left\{L_N, y_N'^\Delta + \frac{\Delta_N}{2}\right\}} P_\delta(\mathbf{y}'|\mathbf{y}^{A\Delta}) dy_1' dy_2' \cdots dy_N' \tag{3-25}$$




(a) (b) (c)

**Fig. 3.** (a) Line charts between the cost and the inspection interval $\delta$. (b) The optimal cubic policy map viewing at position $(1,-1,1)$. (c) The optimal cubic policy map viewing at position $(1,1,1)$.

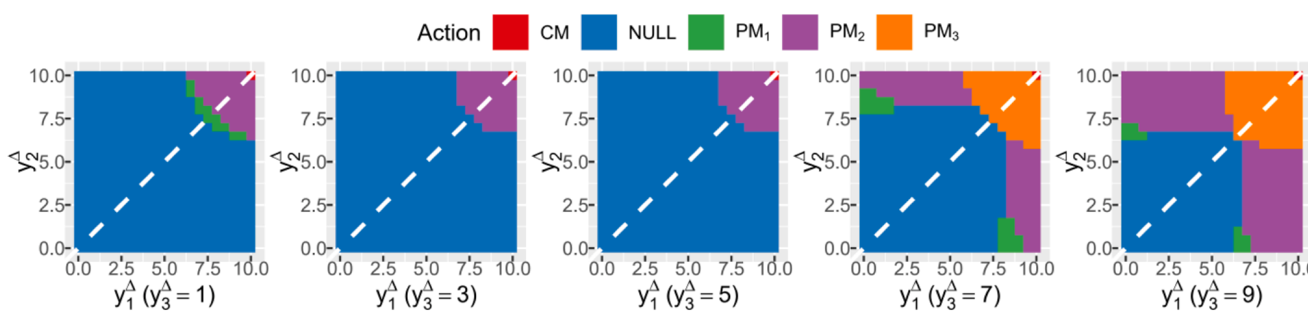



**Fig. 4.** Two-dimensional policy maps of $y_1^{\Delta}$ and $y_2^{\Delta}$ when $y_3^{\Delta}=1,3,5,7,9$.

**Table 2**
Parameter settings for the three stochastic processes and the skewness ($\mathbf{Skew}_i(t)$) and excess kurtosis ($\mathbf{EKurt}_i(t)$) of their degradation increments within time interval $t$.

| Stochastic process | Parameter settings | | $\mathrm{Skew}_i(t)$ | $\mathrm{EKurt}_i(t)$ |
|---|---|---|---|---|
| Wiener process | $\mu_i=1.25$ | $\sigma_i=0.75$ | 0 | 0 |
| Gamma process | $\alpha_i=25/9$ | $\beta_i=20/9$ | $1.20t^{-1/2}$ | $2.16t^{-1}$ |
| IG process | $\nu_i=1.25$ | $\lambda_i=125/36$ | $2.01t^{-1/2}$ | $5.40t^{-1}$ |

where $P_\delta(\mathbf{y}'|\mathbf{y}^{A\Delta})$ is the continuous transition function and like Eq. (3-24), we assign the probability in the small interval $\left[y_i^{'\Delta}-\frac{\Delta_i}{2},y_i^{'\Delta}+\frac{\Delta_i}{2}\right]$ to the discretized point $y_i^{'\Delta}$ for $i=1,2,\ldots,N$.

The expected downtime cost after discretization is the same as Eq. (3-17), and for the consistency of notation, we denote it as $D_\delta^{\Delta}(\mathbf{y}^{A\Delta})$. Therefore, the state-action value function after discretization is

and the Bellman equation becomes

$$V_\delta^{\Delta}(\mathbf{y}^{\Delta})=\begin{cases} Q_\delta^{\Delta}(\mathbf{y}^{\Delta},\mathrm{CM}) & \mathbf{y}^{\Delta}\in S^{F\Delta} \\ \min\{Q_\delta^{\Delta}(\mathbf{y}^{\Delta},\mathrm{NULL}),Q_\delta^{\Delta}(\mathbf{y}^{\Delta},\mathrm{PM}_1),\ldots,Q_\delta^{\Delta}(\mathbf{y}^{\Delta},\mathrm{PM}_N)\} & \text{otherwise}\end{cases} \tag{3-27}$$

After discretization, Monte Carlo simulation is used to estimate the discretized transition function $P_\delta^{\Delta}(\mathbf{y}^{'\Delta}|\mathbf{y}^{A\Delta})$ and the expected downtime cost $D_\delta^{\Delta}(\mathbf{y}^{A\Delta})$. The procedure is summarized in Algorithm 1, where $DP_i(a_i,b_i)$ denotes the cumulative degradation process of component $i$, $i=1,2,\ldots,N$ with parameters $a_i,b_i$ (Wiener process, Gamma process and IG process are three candidates), and algorithms for generating degradation increments characterized by copula function can be found in, for example, Hofert et al. [27].

Based on the transition function and expected downtime cost obtained by Algorithm 1, the value iteration algorithm is performed to find the optimal value of the state value function $V_\delta^{\Delta}(\mathbf{y}^{\Delta})$ and the corresponding policy $\pi_\delta^{\Delta}(\mathbf{y}^{\Delta})$ for any given inspection interval $\delta$, which is

$$Q_\delta^{\Delta}(\mathbf{y}^{\Delta},a)=\begin{cases} c_c+D_\delta^{\Delta}(0)+e^{-r\delta}\sum_{\mathbf{y}^{'\Delta}\in S^{\Delta}}P_\delta^{\Delta}(\mathbf{y}^{'\Delta}|0)V_\delta^{\Delta}(\mathbf{y}^{'\Delta}) & \mathbf{y}^{\Delta}\in S^{F\Delta},a=\mathrm{CM} \\ D_\delta^{\Delta}(\mathbf{y}^{\Delta})+e^{-r\delta}\sum_{\mathbf{y}^{'\Delta}\in S^{\Delta}}P_\delta^{\Delta}(\mathbf{y}^{'\Delta}|\mathbf{y}^{\Delta})V_\delta^{\Delta}(\mathbf{y}^{'\Delta}) & \mathbf{y}^{\Delta}\notin S^{F\Delta},a=\mathrm{NULL} \\ c_s+nc_p+\sum_{\mathbf{y}^{A\Delta}\in S^{\Delta}}P_A^{\Delta}(\mathbf{y}^{A\Delta}|\mathbf{y}^{\Delta},\mathrm{PM}_n)\Big(D_\delta^{\Delta}(\mathbf{y}^{A\Delta})+ & \mathbf{y}^{\Delta}\notin S^{F\Delta},a=\mathrm{PM}_n, \\ \quad e^{-r\delta}\sum_{\mathbf{y}^{'\Delta}\in S^{\Delta}}P_\delta^{\Delta}(\mathbf{y}^{'\Delta}|\mathbf{y}^{A\Delta})V_\delta^{\Delta}(\mathbf{y}^{'\Delta})\Big) & n=1,2,\ldots,N \end{cases} \tag{3-26}$$

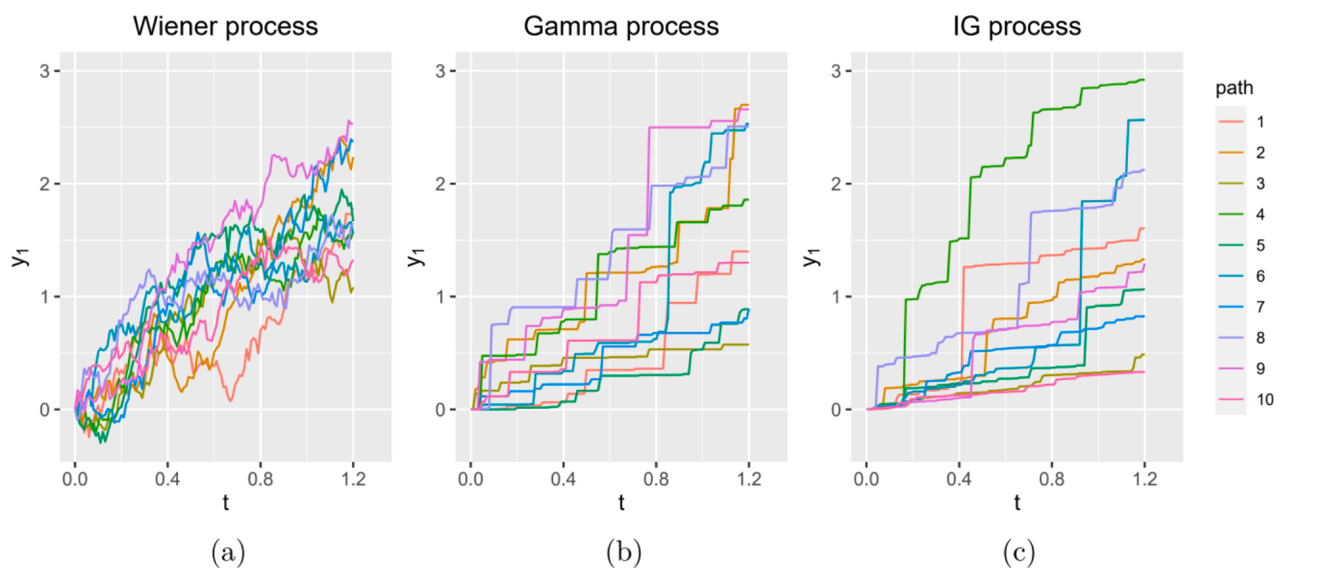



**Fig. 5.** Ten randomly selected degradation paths that evolve from time 0 to time 1.2 for component 1 with degradation subject to (a) Wiener process, (b) Gamma process, and (c) IG process.

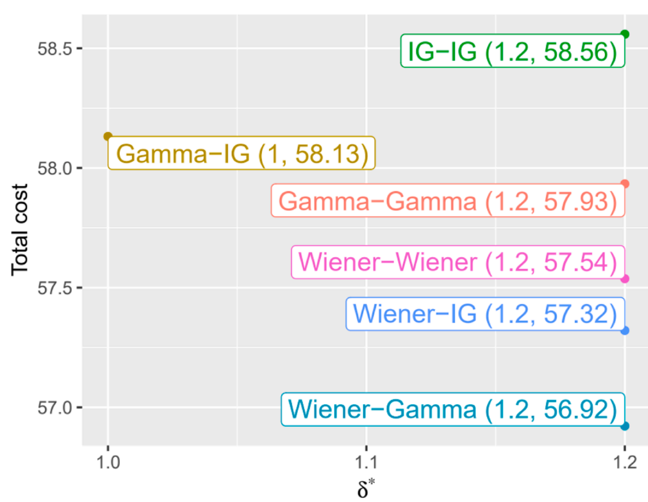



**Fig. 6.** Scatterplot between the optimal inspection interval $\delta^*$ and the corresponding total cost for the six scenarios of the cumulative degradation processes.

summarized in Algorithm 2. Then, the optimal inspection interval is found by

$$\delta^* = \underset{\delta}{\mathrm{argmin}}\left(I_\delta + V_\delta^\Delta(0,0,\ldots,0)\right) \tag{3-28}$$

with optimal CBM policy $\pi_{\delta^*}^\Delta(\mathbf{y}^\Delta)$ obtained by Algorithm 2.

## 4. Numerical Studies and Analysis

### 4.1. Base model

#### 4.1.1. Base model for a 1-out-of-2: G system

We first conduct numerical studies for a 1-out-of-2: G system, for which the CBM policy can be visualized in a two-dimensional policy map. The parameter settings for the base model are as follows:

- We consider two identical components subject to Gamma degradation processes with shape parameter $\alpha_1 = \alpha_2 = 25/9$ and scale parameter $\beta_1 = \beta_2 = 20/9$, so that the mean and variance of the degradation increments within time interval $t$ are $\text{Mean}_i(t) = \alpha_i\beta_i^{-1}t = 1.25t$ (thus the mean degradation rate is $dr_i = 1.25$) and $\text{Variance}_i(t) = \alpha_i\beta_i^{-2}t = (0.75)^2t,\ i = 1,2$, respectively, which is the same as that of Sun et al. [25]. The failure thresholds are set as $L_1 = L_2 = 10$.
- Clayton copula with parameter $\theta = 2$ (Kendall's tau $\tau = 0.5$) is used to model the dependence among the degradation increments with sample interval $h = 0.01$ in Algorithm 1.

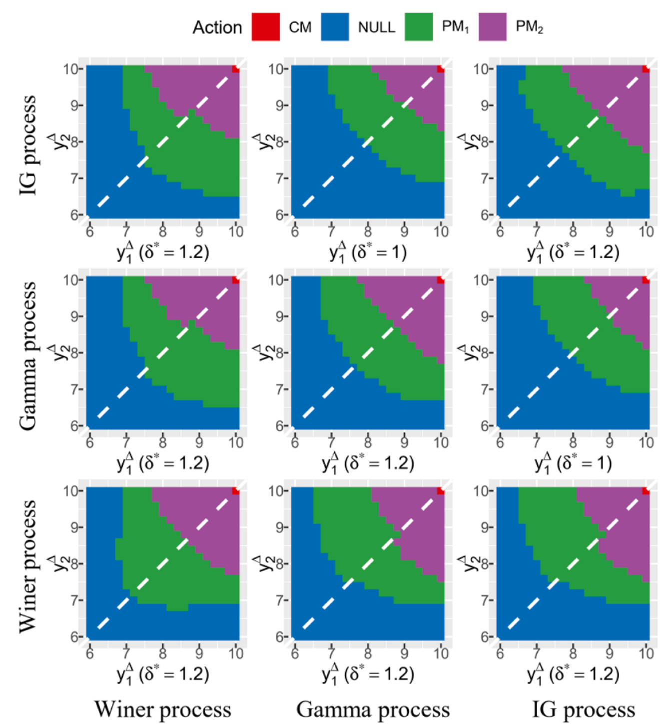



**Fig. 7.** Policy maps of the CBM policy under the optimal inspection interval $\delta^*$ for different scenarios of the degradation processes. Note that for the ease of comparison, the 9 policy maps are incomplete, with the degradation levels ranging from 6 to 10 instead of from 0 to 10. The undrawn regions all correspond to the action **NULL**.

**Table 3**
Parameter settings for the sensitivity analysis on the mean ($\mathbf{Mean}_i(t)$) and variance ($\mathbf{Variance}_i(t)$).

| level | $\mathrm{Mean}_i(t)$ value | $\alpha_i$ | $\beta_i$ | $\mathrm{Variance}_i(t)$ value | $\alpha_i$ | $\beta_i$ |
|---|---|---|---|---|---|---|
| base | $5/4t$ | 25/9 | 20/9 | $3/4t$ | 25/9 | 20/9 |
| high | $15/8t$ | 25/4 | 10/3 | $9/8t$ | 100/81 | 80/81 |
| low | $5/8t$ | 25/36 | 10/9 | $3/8t$ | 100/9 | 80/9 |

- The parameters for the distribution of imperfect maintenance in Eq. (2-18) are set as $p_1 = p_2 = 0.5$ with Beta distribution $\mathscr{Be}(1,1)$.
- Parameters related to cost are as follows: inspection cost $c_i = 0.1$, CM cost $c_c = 15$, downtime cost per time unit $c_d = 2$, shared setup cost for PM $c_s = 1$, PM cost for one component $c_p = 1$. The discount factor is set as $r = 0.01$. The discretization levels are set as $\Delta_1 = \Delta_2 = 0.2$.
- For Algorithm 1, the number of Monte Carlo replications is set as $B = 10000$.
- For Algorithm 2, the stopping threshold for the value iteration algorithm is set as $\epsilon = 10^{-6}$.

We run the value iteration algorithm (Algorithm 2) for $\delta = 0.1, 0.2, \ldots, 3.0$. To find the optimal inspection interval, we plot the line charts between the cost and the inspection interval $\delta$ in Fig. 2 (a). As illustrated in Fig. 2 (a), the curve for the total cost (blue line) is unimodal and the total cost is insensitive to the variation of $\delta$ around the optimal inspection interval $\delta^* = 1.2$. When $\delta$ increases from 0.1 to 3.0, the total cost decreases sharply at first and then increases slowly, which is consistent with the findings of Sun et al. [25]. It is worth mentioning that the above observation also holds for later numerical studies.

To investigate the optimal CBM policy, we draw the heat map of the optimal value function in Fig. 2 (b) and the optimal CBM policy map in Fig. 2 (c). As shown in Fig. 2 (b), the value function $V^{\Delta}_{\delta^*}(y^{\Delta}_1, y^{\Delta}_2)$ is symmetric and non-decreasing with respect to $y^{\Delta}_1, y^{\Delta}_2$. From Fig. 2 (c), we find that the region of the same action is symmetric. This is because the degradation processes of the two components are identical in the base model. In particular, when the degradation level of one component is low, doing nothing (NULL, the blue region) is the optimal action. Then, as the degradation levels of both components become larger, performing PM for the component with the worse condition ($\mathrm{PM}_1$, the green region) is the optimal action. When the degradation levels of both components get even larger, performing PM for both components ($\mathrm{PM}_2$, the purple region) is necessary to avoid potential system failure risks. In the following sections, we denote the lower left border of the non-blue region (in the upper right corner) in Fig. 2 (c) as the maintenance thresholds, which indicates the joint thresholds to perform PM for at least one of the two components.

#### 4.1.2. Base model for a 2-out-of-3: G system

To illustrate the application of the proposed CBM framework in general *K*-out-of-*N*: G systems, we investigate the optimal maintenance policy of a 2-out-of-3: G system. The parameter settings are the same as the base model for a 1-out-of-2: G system, except that (1) the inspection cost $c_i$ is increased from 0.1 to 0.15 since the number of inspected components increases from 2 to 3 upon inspection, and (2) the discretization levels are set as $\Delta_1 = \Delta_2 = \Delta_3 = 0.5$ for the ease of computation.

The results of the iteration algorithm (Algorithm 2) for the 2-out-of-3: G system are shown in Fig. 3. The line charts in Fig. 3 (a) show similar tendency as the 1-out-of-2: G system, and the optimal inspection interval is $\delta^* = 1.4$. Fig. 3 (b) and (c) show the cubic policy map of the optimal CBM policy from two different viewpoints, which shows similar trends and symmetry as the 1-out-of-2: G system. The optimal CBM policy has a general symmetry property: the optimal CBM action for state $(y^{\Delta}_1, y^{\Delta}_2, y^{\Delta}_3)$ is the same as the state that is a permutation of $(y^{\Delta}_1, y^{\Delta}_2, y^{\Delta}_3)$. This is because the degradation processes of the three components are identical. For example, the six states marked by black arrows in Fig. 3 (c) form a permutation of $(10, 6.5, 0)$ and their optimal CBM action are all $\mathrm{PM}_1$ (green region). To better understand the optimal CBM policy for the 2-out-of-3: G system, we slice the optimal cubic policy map based on the degradation state of component 3 ($y^{\Delta}_3$). Fig. 4 shows the two-dimensional policy maps when $y^{\Delta}_3 = 1, 3, 5, 7, 9$. We can find that for a fixed value of $y^{\Delta}_3$, as the degradation levels of component 1 and component 2 become larger, more components need to be maintained. As the degradation level of component 3 increases, the region of maintenance (non-blue region) becomes larger and the maintenance

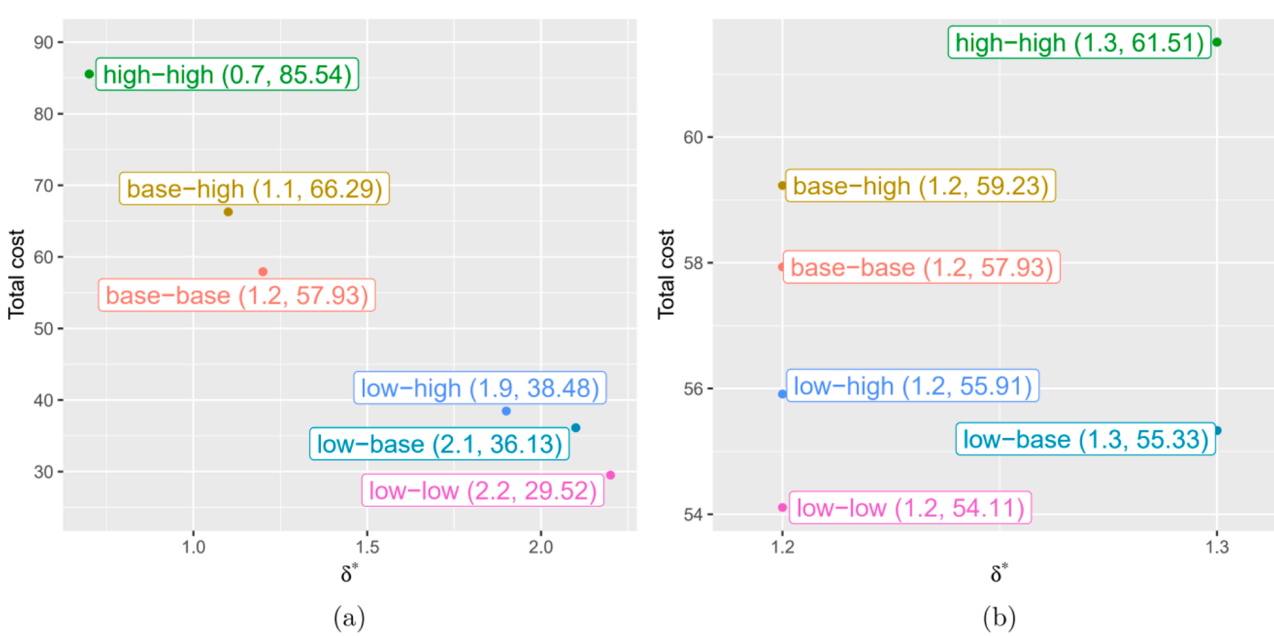



**Fig. 8.** Scatterplots between the optimal inspection interval $\delta^*$ and the corresponding total cost for the six scenarios of (a) the mean and (b) the variance of the degradation increments.

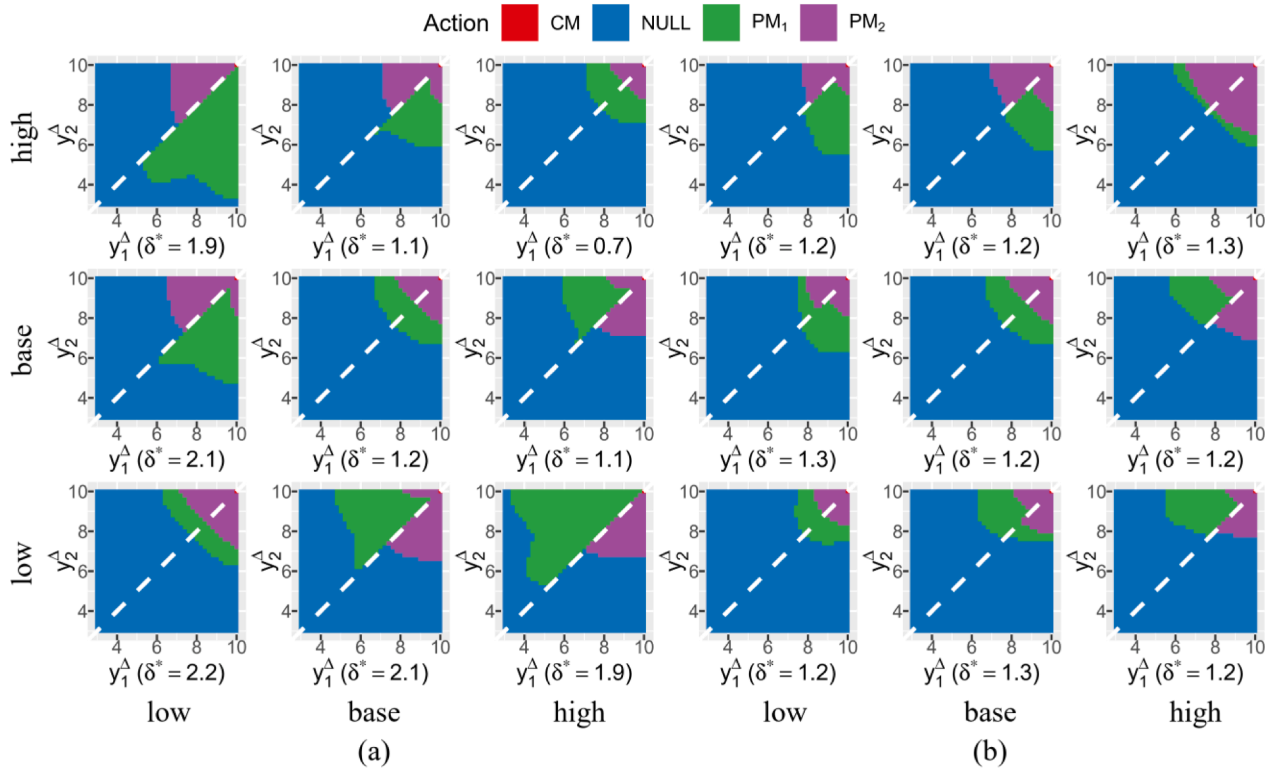



**Fig. 9.** Policy maps of the CBM policy under the optimal inspection interval $\delta^*$ for different scenarios of (a) the mean and (b) variance of the degradation increments. For the ease of comparison, the above 18 policy maps are incomplete, with the degradation levels ranging from 3 to 10 instead of from 0 to 10. The undrawn regions all correspond to the action **NULL**.

thresholds become lower to avoid potential system failure risks.

As the CBM policy for the 1-out-of-2: G system can be visualized in a two-dimensional policy map, which is convenient for analysis, we use the base model for the 1-out-of-2: G system for further investigations.

### *4.2. Impacts of the degradation processes*

To further investigate the impacts of the degradation processes on the optimal maintenance decisions, we extend the types of the two degradation processes (denoted as "$DP_1$-$DP_2$") in the base model for the 1-out-of-2: G system from "Gamma-Gamma" to "Wiener-Wiener", "IG-IG", "Wiener-Gamma" (equivalent to "Gamma-Wiener", the same below), "Wiener-IG" and "Gamma-IG". To make the results of the above six scenarios comparable, the mean and variance are set to be the same for Wiener process, Gamma process and IG process as $\text{Mean}_i(t) = 1.25t$ and $\text{Variance}_i(t) = (0.75)^2 t,\ i = 1, 2$. Table 2 summarizes the related parameter settings for the three stochastic processes as well as the corresponding skewness, denoted as $\text{Skew}_i(t)$, and excess kurtosis, denoted as $\text{EKurt}_i(t)$, of the degradation increments within time interval $t$. Other parameters are the same as the base model.

For each of the three stochastic processes, we randomly select ten degradation paths that evolve from time 0 to time 1.2 and plot them in Fig. 5. We can see that the degradation path of Wiener process is non-monotonic and fluctuates frequently, while those of Gamma process and IG process are monotonic with positive jumps.

To better understand the degradation dependence structure of the six scenarios, we make a detailed analysis by comparing the scatterplots of the degradation increments from time 0 to 0.01 and the scatterplots of the degradation levels at time 1.2 in Supplementary File A. We find that the scenario "Wiener-Wiener" is relatively easier to preserve the dependence through the deterioration cumulative process while others not. This may be explained as follows. On the one hand, we know from Table 2 that the skewness and excess kurtosis of the degradation increments within time interval $t = 0.01$ are both 0 for Wiener process, but they are $\text{Skew}_i(0.01) = 12,\ \text{Ekurt}_i(0.01) = 216$ for Gamma process and $\text{Skew}_i(0.01) = 20,\ \text{Ekurt}_i(0.01) = 540, i = 1, 2$ for IG process. Thus, the distributions of the degradation increments for Gamma process and IG process are positively skewed with heavy right tails. On the other hand, although Clayton copula has lower tail dependence, the dependence in the upper tail is weak. As a result, for scenarios other than "Wiener-Wiener", the relatively larger degradation increments that distribute in the heavy right tails have little dependence and gradually weaken the lower tail dependence through the deterioration cumulative process.

We run the value iteration algorithm (Algorithm 2) for $\delta = 0.1, 0.2, \ldots, 3.0$ for each of the six scenarios. Fig. 6 shows the scatterplot between the optimal inspection interval $\delta^*$ and the corresponding total cost. We can find that $\delta^*$ is insensitive to the types of the cumulative degradation processes, as $\delta^* = 1.2$ for all scenarios except for "Gamma-IG". This is probably because the mean degradation rates of all scenarios are the same.

For the total cost under $\delta^*$, if the two degradation processes are identical (i.e., $DP_1 = DP_2$), the total cost increases in the order of "Wiener-Wiener", "Gamma-Gamma" and "IG-IG". As shown in Table 2, the distribution of the degradation increments for Gamma process and IG process are positively skewed with heavy right tails, and the skewness and excess kurtosis for IG process are larger than Gamma process. Thus, for any initial degradation states, the component (and further the system) subject to IG degradation process is more likely to fail upon inspection than Gamma process and results in a higher maintenance cost. The comparison between "Wiener-Wiener" and "Gamma-Gamma" are similar. If the two degradation processes are heterogeneous (i.e., $DP_1 \neq DP_2$), it shows that the total cost for "$DP_1$-$DP_2$" is smaller than the maximum of that of "$DP_1$-$DP_1$" and "$DP_2$-$DP_2$", but not necessarily bigger than their minimum. For example, the total cost for "Wiener-IG" is smaller than both "Wiener-Wiener" and "Gamma-Gamma", as shown

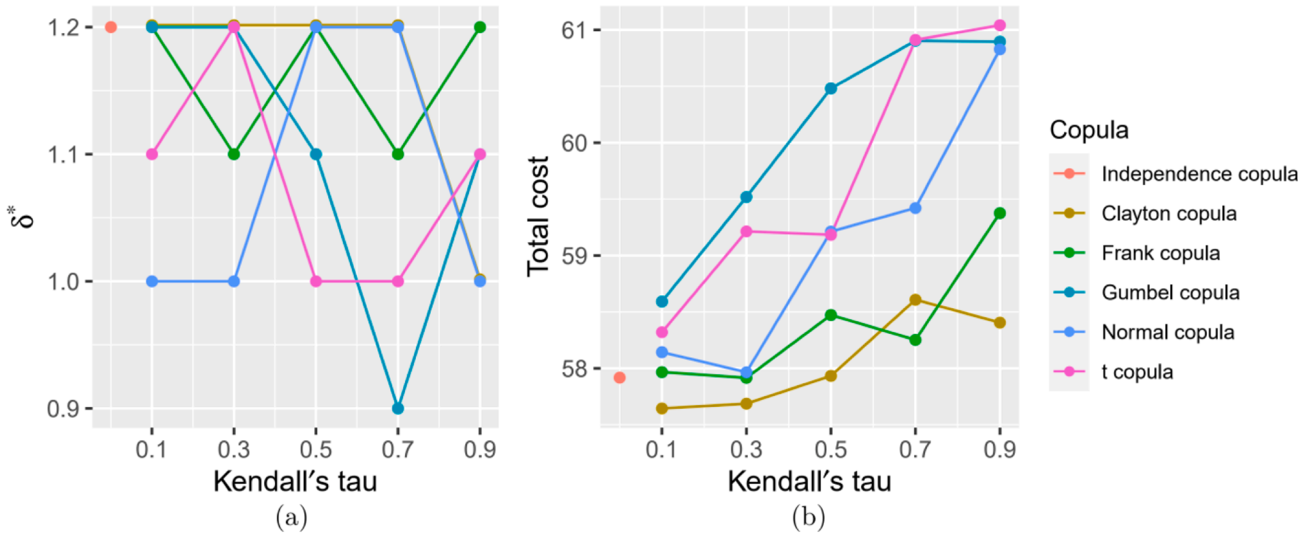



**Fig. 10.** (a) Line charts of the optimal inspection interval $\delta^*$ with respect to Kendall's tau for the five copulas. (b) Line charts of the total cost under $\delta^*$ with respect to Kendall's tau for the five copulas. Note that the red point represents the case of independence copula and is plotted on the figure for reference.

in Fig. 6.

Fig. 7 shows the optimal CBM policy maps for all scenarios in a 3-by-3 grid. If the degradation processes of the two components are identical ("Wiener-Wiener", "Gamma-Gamma", "IG-IG"), we can find that the maintenance thresholds for "Wiener-Wiener" are less dependent (square-like thresholds). This is because there exists strong dependence between the two components for the scenario "Wiener-Wiener" through the deterioration cumulative process, as shown in Supplementary File A. Hence, the deterioration of one component has a significant influence on the other component. It is profitable to immediately maintain the component when the acceleration of one deterioration is detected. **In the context of 1-out-of-2: G system, if the degradation processes of the two components are heterogeneous ("Wiener-Gamma", "Wiener-IG" and "Gamma-IG"), we speculate it is more critical to maintain the component subject to Wiener degradation process.** This can be seen from the scenarios of "Wiener-IG" and "Wiener-Gamma" (the first and second policy maps in the first column of Fig. 7), where the region of maintaining the Wiener degraded component (the green region in the lower triangular plus the purple region) is larger than the other, which indicates that the Wiener degraded component is more critical. This is because for any initial degradation states, the component subject to Wiener degradation process is less likely to fail upon inspection than the heavy-tailed Gamma process and IG process. It is cost-effective to perform PM on the Wiener degraded component with relatively lower thresholds when its degradation level is higher than the other component.

Note that the mean ($\text{Mean}_i(t)$) and variance ($\text{Variance}_i(t)$) of the degradation increments can represent the mean degradation rate and degradation uncertainty, respectively, which can determine the degradation process to some extent. We further perform sensitivity analysis on $\text{Mean}_i(t)$ and $\text{Variance}_i(t)$ separately for the base model to examine their impacts on the optimal maintenance decisions. Table 3 summarizes the values for the base, high and low level of $\text{Mean}_i(t)$ and $\text{Variance}_i(t)$, together with the resulting parameters ($\alpha_i$ and $\beta_i$) of the Gamma process. Apart from the case that the two degradation processes are identical, we also consider the heterogeneous cases. Therefore, for the mean (variance) of the two components (denote "$\text{lvl}_1$-$\text{lvl}_2$" as the parameter levels of the two components respectively), 6 scenarios are considered, including "low-low", "base-base", "high-high", "low-base" (equivalent to "base-low", the same below), "low-high" and "base-high".

Fig. 8 (a) shows the scatterplots between the optimal inspection interval $\delta^*$ and the total cost for different scenarios of the mean. We can find that the total cost increases in the order of "low-low", "low-base", "low-high", "base-base", "base-high" and "high-high", while the optimal inspection interval $\delta^*$ decreases in the same order. Generally, if a system degrades faster (note that a component with larger $\text{Mean}_i(t)$ degrades faster), more frequent inspection is required to prevent potential system failure risks. Meanwhile, the underlying total cost often gets larger due to the larger inspection cost and increasing system failures. Therefore, the above results can be interpreted as follows:

(1) For scenarios "$\text{lvl}_1$-$\text{lvl}_1$" and "$\text{lvl}_2$-$\text{lvl}_2$", if $\text{lvl}_1$ is larger than $\text{lvl}_2$, the system with "$\text{lvl}_1$-$\text{lvl}_1$" will degrade faster than the other, and thus we can speculate that the total cost (the optimal inspection interval $\delta^*$) for "$\text{lvl}_1$-$\text{lvl}_1$" is larger (smaller) than that of "$\text{lvl}_2$-$\text{lvl}_2$".
(2) For scenarios "$\text{lvl}_1$-$\text{lvl}_2$" and "$\text{lvl}_1$-$\text{lvl}_3$" (equivalent to "$\text{lvl}_3$-$\text{lvl}_1$"), if $\text{lvl}_2$ is larger than $\text{lvl}_3$, the system with "$\text{lvl}_1$-$\text{lvl}_2$" will degrade faster than the other, and thus we can speculate that the total cost (the optimal inspection interval $\delta^*$) for "$\text{lvl}_1$-$\text{lvl}_2$" is larger (smaller) than that of "$\text{lvl}_1$-$\text{lvl}_3$".

The results for different scenarios of the variance are shown in Fig. 8 (b). We find out that the total cost follows the same trend as the mean, which can be adjusted similarly. However, the optimal inspection interval is insensitive to the variance. This is possibly because the mean degradation rates for different scenarios are the same.

The optimal CBM policy maps for different scenarios of the mean and variance are shown in Fig. 9. If the two degradation processes are identical ("low-low", "base-base", "high-high"), with the increase of the mean, one should increase the inspection frequency, while with the increase of the variance, one should tend to use group maintenance (to maintain the two components together). Generally, we can interpret the results from the following two perspectives:

(1) In terms of the system degradation, if the system has a higher probability to fail for any given initial states, the CBM policy can be more active to prevent potential system failure. In this case, the maintenance thresholds can be lower and group maintenance is preferable ($\text{PM}_2$).
(2) In terms of the optimal inspection interval $\delta^*$, if $\delta^*$ is relatively smaller, the CBM policy can be more conservative as we inspect the system more frequently. Therefore, the maintenance thresholds can be higher and maintaining each component individually is preferable ($\text{PM}_1$).

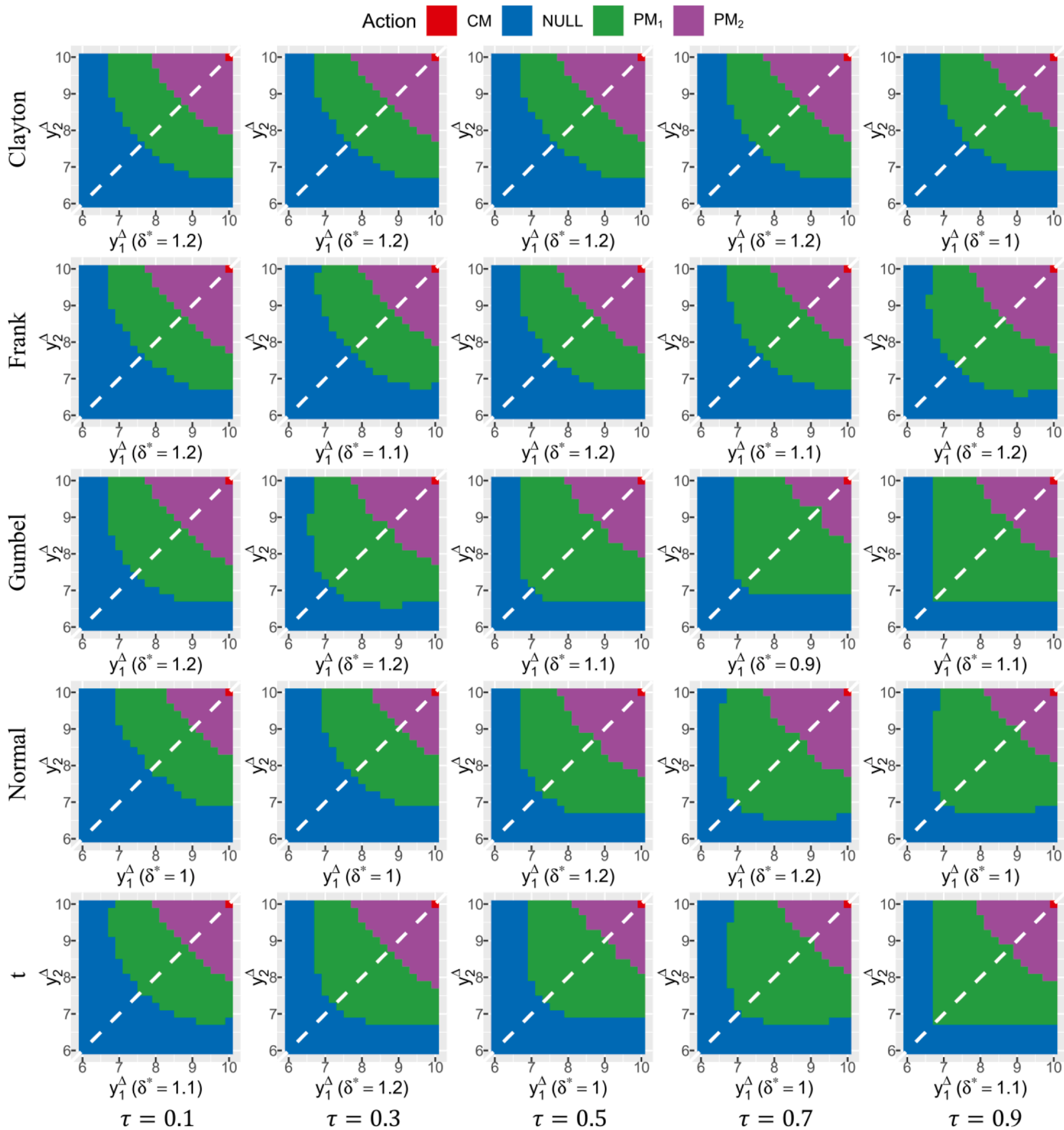



**Fig. 11.** Policy maps of the CBM policy under the optimal inspection interval $\delta^*$ for different copula-Kendall's tau combinations. Each row corresponds to one type of copula and each column corresponds to one level of Kendall's tau. Note that for the ease of comparison, the above 25 policy maps are incomplete, with the degradation levels ranging from 6 to 10 instead of from 0 to 10. The undrawn regions all correspond to the action **NULL**.

As a system with a higher probability to fail usually requires a smaller $\delta^*$, the above two perspectives are not independent, but usually one dominates. Perspective (2) can be used to explain the results for the mean, where $\delta^*$ decreases sharply when the mean changes from low to base to high. Perspective (1) is suitable to explain the results for the variance, where $\delta^*$ is almost unchanged under different scenarios and the system gets more likely to fail with the increase of the variance.

Note that, in the context of 1-out-of-2: G system, if the two degradation processes are heterogeneous ("low-base", "low-high", "base-high"), it is more critical to maintain the component with relatively small mean degradation rate (smaller mean) and uncertainty (smaller variance). Taking the scenario "low-high" in Fig. 9 (a) for example, the mean of component 1 is smaller than component 2. We can find from Fig. 9 (a) that the region of maintaining component 1 (the green region in the lower triangular plus the purple region) is larger than the other, indicating that component 1 is more critical. One possible explanation is that for any initial degradation states, the component with smaller mean (variance) is less likely to fail upon inspection and it is more cost-effective to maintain such component when its degradation level is higher than the other component.

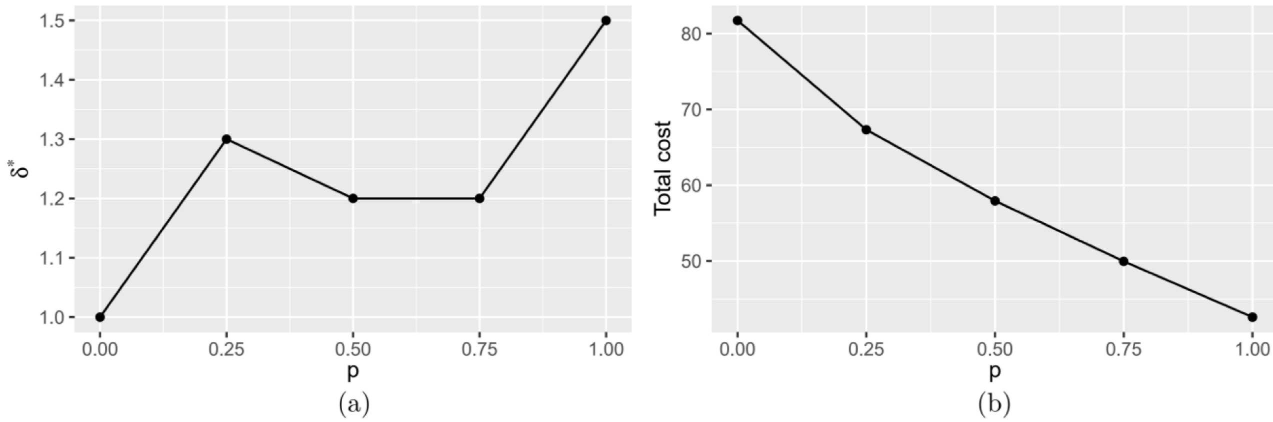



**Fig. 12.** (a) Line chart of the optimal inspection interval $\delta^*$ with respect to $p$. (b) Line chart of the total cost under $\delta^*$ with respect to $p$.

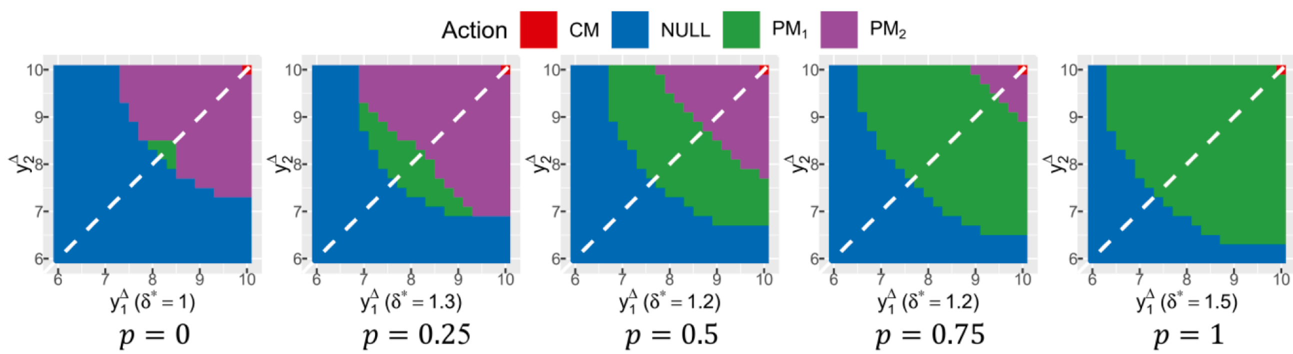



**Fig. 13.** Policy maps of the CBM policy under the optimal inspection interval $\delta^*$ for different levels of $p$. Each column corresponds to one level of $p$. For the ease of comparison, the above 5 policy maps are incomplete with the degradation levels ranging from 6 to 10 instead of from 0 to 10. The undrawn regions all belong to the action **NULL**.

**Table 4**
Parameter settings for sensitivity analysis on the five cost-related parameters.

| | $c_i$ | $c_c$ | $c_d$ | $c_s$ | $c_p$ |
|---|---|---|---|---|---|
| base | 0.1 | 15 | 2 | 1 | 1 |
| high | 0.15 | 22.5 | 3 | 1.5 | 1.5 |
| low | 0.05 | 7.5 | 1 | 0.5 | 0.5 |

### *4.3. Impacts of the copula functions*

To further investigate the impacts of copula functions on the optimal maintenance decisions, we extend the copula function in the base model from Clayton copula to Frank copula, Gumbel copula, normal copula and t copula (with $\nu = 2$ degrees of freedom). For each copula, 5 levels of Kendall's tau ($\tau$) including 0.1, 0.3, 0.5, 0.7, 0.9 are considered to indicate different degrees of association between the two degradation processes. Thus, there are totally 25 scenarios and the other parameters are the same as the base model.

To better understand the degradation dependence structure of the 25 scenarios, we make a detailed analysis by comparing the scatterplots of the degradation increments from time 0 to 0.01 and the scatterplots of the degradation levels at time 1.2 in Supplementary File B. We find that for each of the five copulas, as Kendall's tau increases, the positive association between the degradation increments of the two components becomes stronger. Meanwhile, the dependence between the two components is easier to preserve through the deterioration cumulative process for Gumbel copula, normal copula and t copula, but is hard to preserve for Clayton copula and Frank copula. This is possibly because that Gumbel copula, normal copula and t copula are three copulas with upper tail dependence. Besides, for copulas without upper tail dependence, such as Clayton copula and Frank copula, the dependence may be easier to preserve if the two degradation processes are both Wiener process, as what we have discussed in Section 4.2.

Again, we run the value iteration algorithm for $\delta = 0.1, 0.2, \ldots, 3.0$ for each of the 25 scenarios, and we also consider the scenario with two independent cumulative degradation processes by using the independence copula. We draw the line charts of the optimal inspection interval $\delta^*$ with respect to Kendall's tau for the five copulas in Fig. 10 (a). We can see that $\delta^*$ fluctuates between 0.9 and 1.2. Combined with the results in Section 4.1 that $\delta$ is robust around the optimal value, we may conclude that $\delta^*$ is insensitive to both the types of copula and Kendall's tau. Fig. 10 (b) shows the line charts of the total cost under $\delta^*$, from which we can observe that for all 5 copulas, the total cost tends to be larger when Kendall's tau increases. On average, the total cost increases in the order of Clayton copula, Frank copula, normal copula, t copula and Gumbel copula. Note that the first three copulas are with upper tail dependence and is easier to preserve the dependence through the deterioration cumulative process while the last two copulas not, as discussed at the beginning of this section. We may speculate that the total cost will be larger if the dependence among the two components is easier to preserve, under which condition there exists strong positive association between the degradation levels of the two components. Thus, the system fails more frequently and the maintenance cost will be larger.

Fig. 11 shows the optimal CBM policy maps for all 25 scenarios, arranged in a 5-by-5 grid. **Remarkably, it seems that the component-specific maintenance thresholds dependence is inversely related to the degradation dependence of the two components.** For each copula, the more dependence between the two components (the larger the Kendall's tau), the less dependence in the thresholds for $PM_1$ (square like thresholds). One possible explanation is that with larger Kendall's

**Algorithm 1**

Algorithm for estimating $P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta})$ and $D_\delta^\Delta(\mathbf{y}^{A\Delta})$

**Input**: Cumulative degradation processes: $DP_i(a_i, b_i)$. Failure threshold: $L_i$. Sample interval of the degradation path: $h$. Copula function: $\mathscr{C}_h$. Inspection interval: $\delta$. Downtime cost: $c_d$. Discount factor: $r$. Discretization level: $\Delta_i$. Initial degradation state: $\mathbf{y}^{A\Delta} = [y_1^{A\Delta}, y_2^{A\Delta}, \ldots, y_N^{A\Delta}]$. Number of replications: $B$.

**Output**: Estimated $P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta})$ for all $\mathbf{y}'^\Delta \in S^\Delta$ and $D_\delta^\Delta(\mathbf{y}^{A\Delta})$.

**begin**

Initialization: Set $P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta}) = 0$ for all $\mathbf{y}'^\Delta \in S^\Delta$, and $D_\delta^\Delta(\mathbf{y}^{A\Delta}) = 0$;

**for** $b = 1, 2, \ldots, B$

Set $y_{i,0}^b = y_i^{A\Delta}, i = 1, 2, \ldots, N$; // Initialize the initial states for the degradation paths

**for** $t = 1, h, \ldots, \left\lfloor \frac{\delta}{h} \right\rfloor h$

Generate degradation increments $\Delta y_{i,t}^b, i = 1, 2, \ldots, N$ with cumulative degradation processes $DP_i(a_i, b_i)$ and copula function $C_h$; // See Hofert et al. [27] for related algorithms

Set $y_{i,t}^b = y_{i,t-1}^b + \Delta y_{i,t}^b, i = 1, 2, \ldots, N$;

**end for**

**for** $i = 1, 2, \ldots, N$

Set $y_i'^b = y^b_{i,\left\lfloor \frac{\delta}{h} \right\rfloor h}$; // Compute the state upon the next inspection, denoted as $y_i'^b$

Compute the failure time of component $i$, denoted as $t_i^b$ (set as Inf if not fails over $[0, \delta]$);

**if** $t_i^b <$ Inf **then**

Set $y_i'^b = L_i$; // The state upon the next inspection is $L_i$ if component $i$ fails over $[0, \delta]$

**end if**

**end for**

Compute the failure time of the system, denoted as $t^b$ (set as Inf if not fails over $[0, \delta]$);

**if** $t^b <$ Inf **then**

Set $D_\delta^\Delta(\mathbf{y}^{A\Delta}) = D_\delta^\Delta(\mathbf{y}^{A\Delta}) + \frac{c_d(e^{-rt^b} - e^{-r\delta})}{r}$;

**end if**

Set $y_i'^{\Delta b} = \max\left\{\left\lfloor \frac{y_i'^b}{\Delta_i} + \frac{1}{2} \right\rfloor, 0\right\}$, $i = 1, 2, \ldots, N$; // Compute the discretized transitioned states

Set $P_\delta^\Delta(\mathbf{y}'^{\Delta b}|\mathbf{y}^{A\Delta}) = P_\delta^\Delta(\mathbf{y}'^{\Delta b}|\mathbf{y}^{A\Delta}) + 1$; // $\mathbf{y}'^{\Delta b} = [y_1'^{\Delta b}, y_2'^{\Delta b}, \ldots, y_N'^{\Delta b}]$

**end for**

Set $D_\delta^\Delta(\mathbf{y}^{A\Delta}) = D_\delta^\Delta(\mathbf{y}^{A\Delta})/B$; // Compute the expectation by dividing the number of replications

Set $P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta}) = P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta})/B$ for all $\mathbf{y}'^\Delta \in S^\Delta$; // Transform frequency into probability

**end**

tau, the deterioration of one component has a significant influence on the other. Therefore, it is profitable to immediately maintain the component when the acceleration of one deterioration is detected. Hence, the maintenance thresholds for $PM_1$ are relatively less dependent. In extreme situation, the maintenance thresholds for $PM_1$ are like a square, such as the scenarios "Gumbel-0.9" and "t-0.9" in Fig. 11, where PM is performed on the worse component when the degradation levels of both components exceed the thresholds. **One observed trend is that the component-specific maintenance thresholds are less dependent for components with upper tail degradation dependence.** For a specific Kendall's tau, the thresholds for $PM_1$ is less dependent for copulas with upper tail dependence (e.g., Gumbel copula, normal copula and t copula) than copulas without upper tail dependence (e.g., Clayton copula, Frank copula). We have analyzed previously that for copulas with upper tail dependence, the dependence between the two components is easier to preserve through the deterioration cumulative process. Thus, the deterioration of one component has a more significant influence on the other for copulas with upper tail dependence than without. As a result, it is more profitable to immediately maintain the component when the acceleration of one deterioration is detected and the thresholds for $PM_1$ is less dependent.

**Algorithm 2**

Value iteration algorithm for estimating $V_\delta^\Delta(\mathbf{y}^\Delta)$ and $\pi_\delta^\Delta(\mathbf{y}^\Delta)$

**Input**: Cumulative degradation processes: $DP_i(a_i, b_i)$. Failure threshold: $L_i$. Sample interval of the degradation path: $h$. Copula function: $\mathscr{C}_h$. Imperfect maintenance: $p_i$, $\mathscr{B}e(a_i, b_i)$. Inspection interval: $\delta$. Cost parameters: $c_i, c_c, c_d, c_s, c_p$. Discount factor: $r$. Discretization level: $\Delta_i$. Number of replications: $B$. Stopping threshold: $\epsilon$.

**Output**: Estimated state value function $V_\delta^\Delta(\mathbf{y}^\Delta)$ and corresponding policy $\pi^\Delta(\mathbf{y}^\Delta)$ for all $\mathbf{y}^\Delta \in S^\Delta$.

**begin**

Initialization: Set $V_\delta^\Delta(\mathbf{y}^\Delta) = 0, V_\delta^{\Delta(1)}(\mathbf{y}^\Delta) = \epsilon, \pi_\delta^\Delta(\mathbf{y}^\Delta) = -1$ for all $\mathbf{y}^\Delta \in S^\Delta$;

Compute $P_A^\Delta(\mathbf{y}^{A\Delta}|\mathbf{y}^\Delta, a)$ for all $\mathbf{y}^\Delta, \mathbf{y}^{A\Delta} \in S^\Delta$, $a \in A(\mathbf{y}^\Delta)$ by Eq. (3-24);

Compute $P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta})$ and $D_\delta^\Delta(\mathbf{y}^{A\Delta})$ for all $\mathbf{y}'^\Delta, \mathbf{y}^{A\Delta} \in S^\Delta$ using Algorithm 1;

**while** $\max_{\mathbf{y}^\Delta \in S^\Delta} |V_\delta^{\Delta(1)}(\mathbf{y}^\Delta) - V_\delta^\Delta(\mathbf{y}^\Delta)| \geq \epsilon$ **do**

Set $V_\delta^\Delta(\mathbf{y}^\Delta) \leftarrow V_\delta^{\Delta(1)}(\mathbf{y}^\Delta)$;

**for** $\mathbf{y}^\Delta \in S^\Delta$

**if** $\mathbf{y}^\Delta \in S^{F\Delta}$ **then** // $S^{F\Delta} \subset S^\Delta$ is the set of system failure states

Set $V_\delta^{\Delta(1)}(\mathbf{y}^\Delta) \leftarrow c_c + D_\delta^\Delta(0) + e^{-r\delta} \sum_{\mathbf{y}'^\Delta \in S^\Delta} P_\delta^\Delta(\mathbf{y}'^\Delta|0) V_\delta^\Delta(\mathbf{y}'^\Delta)$;

Set $\pi_\delta^\Delta(\mathbf{y}^\Delta) \leftarrow$ CM;

**else**

Compute:

$$Q_\delta(\mathbf{y}^\Delta, \text{NULL}) = D_\delta^\Delta(\mathbf{y}^\Delta) + e^{-r\delta} \sum_{\mathbf{y}'^\Delta \in S^\Delta} P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^\Delta) V_\delta^\Delta(\mathbf{y}'^\Delta);$$

$$Q_\delta(\mathbf{y}^\Delta, \text{PM}_n) = c_s + nc_p + \sum_{\mathbf{y}^{A\Delta} \in S^\Delta} P_A^\Delta(\mathbf{y}^{A\Delta}|\mathbf{y}^\Delta, \text{PM}_n)(D_\delta^\Delta(\mathbf{y}^{A\Delta}) + \sum$$

$$e^{-r\delta} \sum_{\mathbf{y}'^\Delta \in S^\Delta} P_\delta^\Delta(\mathbf{y}'^\Delta|\mathbf{y}^{A\Delta}) V_\delta^\Delta(\mathbf{y}'^\Delta)), n = 1, 2, \ldots, N;$$

Set $V_\delta^{\Delta(1)}(\mathbf{y}^\Delta) \leftarrow \min_a Q_\delta^\Delta(\mathbf{y}^\Delta, a)$;

Set $\pi_\delta^\Delta(\mathbf{y}^\Delta) \leftarrow \operatorname{argmin}_a Q_\delta^\Delta(\mathbf{y}^\Delta, a)$; // Choose the optimal action

**end if**

**end for**

**end while**

**end**

### 4.4. Impacts of the imperfect maintenance

Recall that in Section 2.4 we assume that there is a reduction $PM_{i,y_i^-}$ in the degradation level after performing PM on component $i$ for $i = 1, 2$, and the cdf of $PM_{i,y_i^-}$ is

$$F_{PM_{i,y_i^-}}(y) = \begin{cases} (1-p_i) F_{\mathscr{B}e}\left(\frac{y}{y_i^-}; a_i, b_i\right) & 0 \leq y < y_i^- \\ 1 & y = y_i^- \end{cases} \tag{4-1}$$

where $y_i^-$ is the degradation level of component $i$ immediately before PM, $p_i \in [0, 1]$ is the probability that the component after PM is as-good-as-new, and $1 - p_i$ can reflect the degree of imperfect maintenance.

To further explore the impacts of imperfect maintenance on the optimal maintenance decisions, we focus on the effects of $p_i$, $i = 1, 2$ and explore the case when $p_1 = p_2 = p$. Five levels of $p$ are considered, including 0, 0.25, 0.5, 0.75, 1, where $p = 1$ corresponds to perfect maintenance. The other parameters are the same as the base model.

We run the value iteration algorithm for $\delta = 0.1, 0.2, \ldots, 3.0$ for each level of $p$. The line charts for the optimal inspection interval $\delta^*$ and the total cost under $\delta^*$ are shown in Fig. 12. It shows that with the increase of the degree of imperfect maintenance (the decrease of $p$), there is a decreasing trend for the optimal inspection interval $\delta^*$ and there is an increasing trend for the total cost. One explanation is that due to the imperfect maintenance, the maintenance action is less effective than before. Therefore, the inspection interval is reduced to prevent system failure risks and both the inspection cost and maintenance cost will be higher.

The optimal CBM policy maps are shown in Fig. 13. We speculate that if the maintenance is effective, the system is prone to individual maintenance; if the maintenance is less effective, the group maintenance is preferable. We can interpret this remark as follows. When $p$ decreases

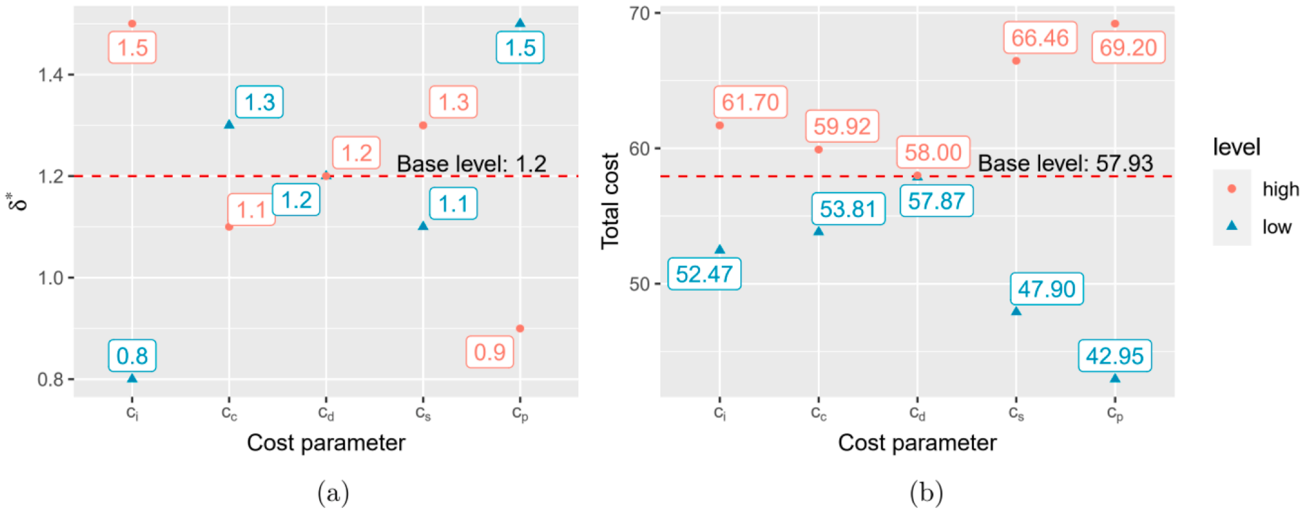



**Fig. 14.** Scatterplots for (a) the optimal inspection interval $\delta^*$ and (b) the total cost with respect to the five cost parameters. The horizontal red dashed line corresponds to the result of the base model.

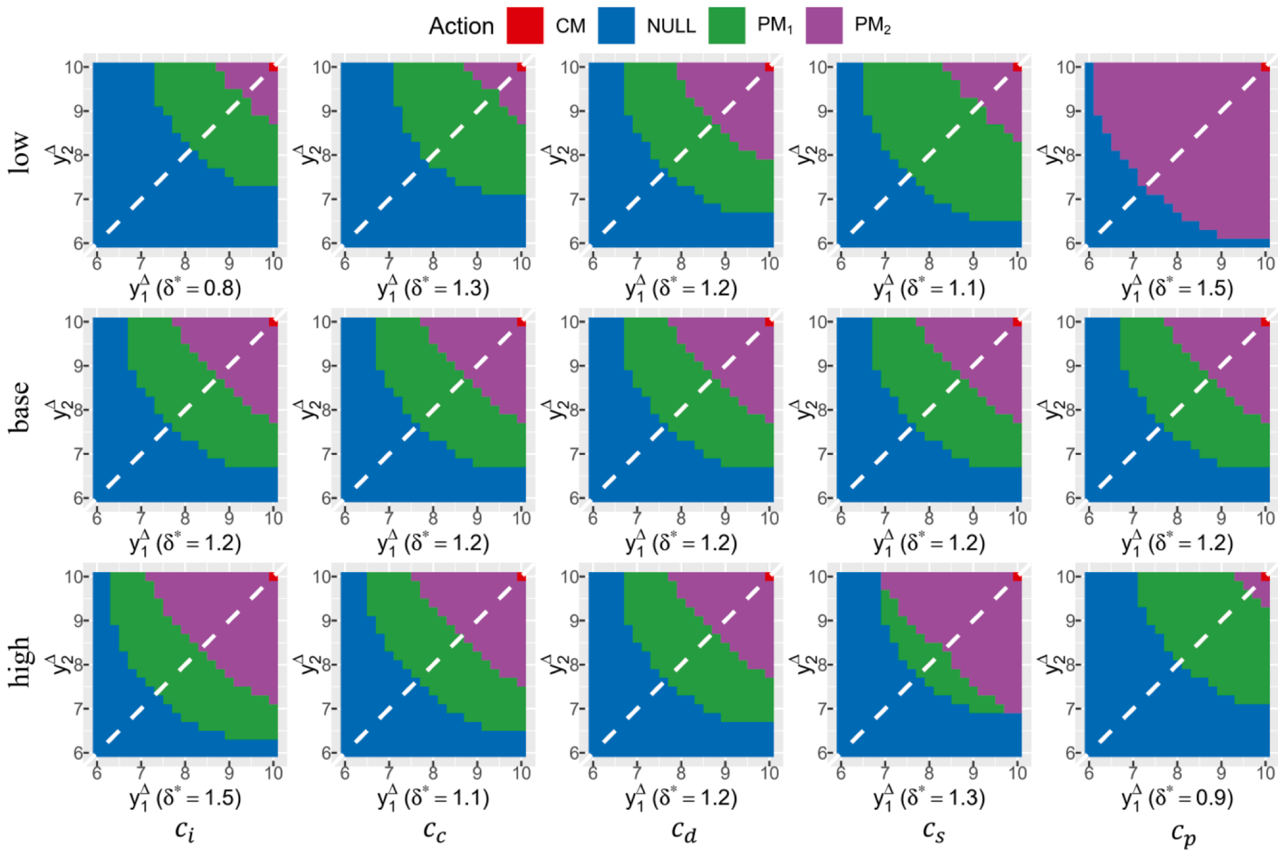



**Fig. 15.** Policy maps of the CBM policy under the optimal inspection interval $\delta^*$ for the sensitivity analysis on the five cost parameters. Each row corresponds to one level of parameter setting and each column corresponds to one cost parameter. Note that for the ease of comparison, the above 15 policy maps are incomplete, with the degradation levels ranging from 6 to 10 instead of from 0 to 10. The undrawn regions all correspond to the action **NULL**.

from 1 to 0 (from the right to the left in Fig. 13), the effectiveness of the maintenance gets lower. With less effective maintenance, the component after PM tends to have a worse condition. Therefore, we need to perform PM on the two components together more to compensate for the effect of imperfect maintenance. Besides, when the maintenance is less effective, we tend to inspect the system more frequently. Thus, it is sufficient to perform the maintenance action when the degradation levels of the two components are high enough.

### *4.5. Sensitivity analysis for a 1-out-of-2: G system*

Sensitivity analysis on the remaining five cost-related parameters are conducted in this section to investigate their impacts on the optimal maintenance decisions, which is similar to that in Sun et al. [25] and we also get some comparable results. Table 4 summarizes the parameter settings for the base (parameter settings in the base model), high (50% higher than the base model) and low (50% lower than the base model) levels of the five cost parameters, and the other parameters are unchanged compared with the base model.

We run the value iteration algorithm for $\delta = 0.1, 0.2, \ldots, 3.0$ for each scenario. Fig. 14 shows the scatterplots for the optimal inspection interval $\delta^*$ and the corresponding total cost with respect to the five cost parameters, where the horizontal red dashed line corresponds to the result of the base model. The optimal policy maps are shown in Fig. 15. From Fig. 14 (a), we can find that $\delta^*$ is highly sensitive to $c_i$ and $c_p$, slightly sensitive to $c_c$ and $c_s$, but not sensitive to $c_d$. It shows from Fig. 14 (b) that the total cost under $\delta^*$ is highly sensitive to $c_s$ and $c_p$, slightly sensitive to $c_i$ and $c_c$, but not sensitive to $c_d$ as well. And all the total cost in the high (low) level of the five cost parameters are larger (smaller) than the base level.

Specifically, when the inspection cost $c_i$ increases, the periodic inspection should be less frequent due to the cost consideration. Meanwhile, according to the policy maps in the first column of Fig. 15, the maintenance thresholds get lower and maintaining the two components together is preferable. For the shared setup cost $c_s$, it exhibits characteristics similar to those of $c_i$. In contrast to $c_i$, when the CM cost $c_c$ or the PM cost $c_p$ increases, more frequent inspection is required to lessen the cost caused by system failure. However, $c_c$ and $c_p$ show opposite trends in the policy maps shown in Fig. 15. In particular, $c_c$ behaves like $c_i$, while when $c_p$ increases, the maintenance thresholds get higher and maintaining each component individually is preferable. Finally, the down time cost $c_d$ is insensitive to both the optimal inspection interval $\delta^*$ and the total cost, which is also confirmed by the results that the three policy maps under different levels of $c_d$ are very similar to each other, as shown in the third column of Fig. 15.

## 5. Conclusions

With the development of sensor and communication technology, CBM plays an important role in system maintenance, especially for multi-component systems. In this study, we investigate the generalized CBM optimization problem for multi-component systems by emphasizing the stochastic dependency among components and considering imperfect maintenance. The system under study is a $K$-out-of-$N$: G system with $N$ dependent components under periodic inspection. Our objective is to minimize the expected long-run discounted cost. In the model, the cumulative degradation of an individual component is modeled by heterogeneous stochastic processes. Copula function is used to characterize the degradation dependence, and the imperfect maintenance is represented by a reduction in the degradation level. Due to the Markov property of the proposed CBM model, MDP is used to solve the problem, and the state, action, policy, state transition function and reward of the MDP model are well defined. To ease the computation burden, we discretize the continuous state space and then use the value iteration algorithm with Monte Carlo simulation to find the optimal inspection interval and the corresponding CBM policy.

After investigating the 1-out-of-2: G system, we conclude four remarks that might be beneficial for guiding the maintenance in practice.

- **Impacts of the types of degradation processes:** Based on the numerical testing, it seems that it is more critical to preserve the component subject to Wiener degradation process if the two degradation processes are "Wiener-Gamma" or "Wiener-IG" with the same mean degradation rates.
- **Impacts of the mean degradation rate and uncertainty:** It seems that for a system with two Gamma degradation processes, if the two degradation processes are identical, with the increase of the mean degradation rate, one should increase the inspection frequency, while with the increase of the degradation uncertainty, one should tend to use group maintenance (to maintain the two components together); If the two degradation processes are heterogeneous, it is more critical to preserve the component with relatively low mean degradation rate and uncertainty.
- **Impacts of the copula functions:** We speculate that for a system with two identical Gamma degradation processes, the component-specific maintenance thresholds dependence is inversely related to the degradation dependence of the two components, and the component-specific maintenance thresholds are less dependent for components with upper tail degradation dependence.
- **Impacts of the imperfect maintenance:** We speculate that for a system with two identical Gamma degradation processes, if the maintenance is effective, the system is prone to individual maintenance; If the maintenance is less effective, the group maintenance is preferable.

For future research, the above conclusive remarks obtained from a 1-out-of-2: G system can be generalized to the $K$-out-of-$N$: G system with $N \geq 3$, for which the maintenance decisions will be more diverse. Although some results are intuitively convincing and verifiable by numerical testings, further studies such as rigorous derivations are still recommended. Moreover, when ranking the failure extent of $N$ components, although the failure threshold-normalized degradation level is easier to implement in practice, using the expected remaining useful life is another reasonable alternative, which deserves further investigation.

## CRediT authorship contribution statement

**Jun Xu:** Writing – original draft, Writing – review & editing, Conceptualization, Methodology. **Zhenglin Liang:** Writing – original draft, Writing – review & editing, Conceptualization, Methodology. **Yan-Fu Li:** Supervision, Writing – review & editing, Conceptualization, Methodology. **Kaibo Wang:** Supervision, Writing – review & editing, Conceptualization, Methodology.

## Declaration of Competing Interest

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted. We have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

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## Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ress.2021.107592.

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